

AN $L_q(L_p)$ -THEORY FOR THE TIME FRACTIONAL EVOLUTION EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT. We introduce an $L_q(L_p)$ -theory for the quasi-linear fractional equations of the type

$$\partial_t^\alpha u(t, x) = a^{ij}(t, x) u_{x^i x^j}(t, x) + f(t, x, u), \quad t > 0, x \in \mathbf{R}^d.$$

Here, $\alpha \in (0, 2)$, $p, q > 1$, and ∂_t^α is the Caputo fractional derivative of order α . Uniqueness, existence, and $L_q(L_p)$ -estimates of solutions are obtained. The leading coefficients $a^{ij}(t, x)$ are assumed to be piecewise continuous in t and uniformly continuous in x . In particular $a^{ij}(t, x)$ are allowed to be discontinuous with respect to the time variable. Our approach is based on classical tools in PDE theories such as the Marcinkiewicz interpolation theorem, the Calderon-Zygmund theorem, and perturbation arguments.

1. INTRODUCTION

Fractional calculus has been used in numerous areas including mathematical modeling [24, 44], control engineering [6, 32], electromagnetism [13, 43], polymer science [3, 29], hydrology [4, 41], biophysics [14, 22], and even finance [36, 40]. See also [15, 30, 37, 48] and references therein. The classical heat equation $\partial_t u = \Delta u$ describes the heat propagation in homogeneous mediums. The time-fractional diffusion equation $\partial_t^\alpha u = \Delta u$, $\alpha \in (0, 1)$, can be used to model the anomalous diffusion exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena (see [25, 28]). The fractional wave equation $\partial_t u = \Delta u$, $\alpha \in (1, 2)$ governs the propagation of mechanical diffusive waves in viscoelastic media (see [23]). The fractional differential equations have another important issue in the probability theory related to non-Markovian diffusion processes with a memory (see [26, 27]).

The main goal of this article is to present an $L_q(L_p)$ -theory for the quasi-linear fractional evolution equation

$$\partial_t^\alpha u(t, x) = a^{ij}(t, x) u_{x^i x^j}(t, x) + b^i(t, x) u_{x^i}(t, x) + c(t, x) u(t, x) + f(t, x, u) \quad (1.1)$$

given for $t > 0$ and $x \in \mathbf{R}^d$. Here $\alpha \in (0, 2)$, $p, q > 1$, and ∂_t^α denotes the Caputo fractional derivative (see (2.3)). The indices i and j move from 1 to d , and the summation convention with respect to the repeated indices is assumed throughout the article. It is assumed that the leading coefficients $a^{ij}(t, x)$ are piecewise continuous in t and uniformly continuous in x , and the lower order coefficients b^i and c are only bounded measurable functions. We prove that under a mild condition

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on the nonlinear term $f(t, x, u)$ there exists a unique solution u to (1.1) and the $L_q(L_p)$ -norms of the derivatives $D_x^\beta u$, $|\beta| \leq 2$, are controlled by the $L_q(L_p)$ -norm of $f(t, x, 0)$.

We remark that there are a few other types of fractional derivatives such as Riemann-Liouville, Marchaud, and Grünwald-Letnikov fractional derivatives. These three fractional derivatives coincide with the Caputo fractional derivative in our solution space $\mathbb{H}_{q,p,0}^{\alpha,n}(T)$ (see [39, 31] for the proof).

Here is a brief survey of closely related works. In [8, 33] an $L_q(L_p)$ -theory for the parabolic Volterra equations of the type

$$\frac{\partial}{\partial t} \left(c_0 u + \int_{-\infty}^t k_1(t-s)u(s, x)ds \right) = \Delta u + f(t, x, u), \quad t \in \mathbf{R}, x \in \mathbf{R}^d \quad (1.2)$$

is obtained under the conditions $k_1(t) \geq ct^{-\alpha}$ for small t , $c_0 \geq 0$, $\alpha \in (0, 1)$, and

$$\frac{2}{\alpha q} + \frac{d}{p} < 1. \quad (1.3)$$

The results of [8, 33] also cover the case $c_0 > 0$, however it is obtained only for the case $a^{ij}(t, x) = \delta^{ij}$ with the restrictions $\alpha \in (0, 1)$ and (1.3). If $p = q$, an L_p -theory of type (1.1) with the variable coefficients $a^{ij}(t, x)$ is presented in [46] under the condition that a^{ij} are uniformly continuous in (t, x) and $\lim_{|x| \rightarrow \infty} a^{ij}(t, x)$ exists. In [47] an L_2 -theory is obtained for the divergence type equations with general measurable coefficients. Also an eigenfunction expansion method is introduced in [38] to obtain L_2 -estimates of solutions of divergence type equations with C^1 -coefficients.

For other approaches to the equations with fractional time derivatives, we refer to [21] for the semigroup approach, to [9, 34] for $C^\delta([0, T], X)$ -type theory, where X is an appropriate Banach space, and to [7] for $BUC_{1-\beta}([0, T], X)$ -type estimate, where $\|u\|_{BUC_{1-\beta}([0, T], X)} = \sup_{t \in (0, T]} t^{1-\beta} \|u(t)\|_X$.

Our result substantially generalizes above mentioned results in the sense that we do not impose any algebraic conditions on α , p , and q . The conditions (1.3) and $\alpha \in (0, 1)$ are used in [8, 33], and the restrictions $p = q$ and $\alpha \notin \{\frac{2}{2p-1}, \frac{2}{p-1} - 1, \frac{1}{p}, \frac{3}{2p-1}\}$ are assumed in [46]. More importantly, in this article the condition on the leading coefficients $a^{ij}(t, x)$ is considerably weakened. In particular, $a^{ij}(t, x)$ depend on both t and x and can be discontinuous in t . Recall that if $p > 2$ then among above articles only [46] considers the coefficients depending also on t , but the condition $p = q$ and the continuity of a^{ij} with respect to (t, x) are assumed in [46].

Another significance of this article is the method we use. The results of [8, 33, 46] are operator theoretic, [21] is based on H^∞ -functional calculus, and the method of [47, 38] works well only in the Hilbert-space framework. Our approach is purely analytic and is based on the classical tools in PDE theories including the Marcinkiewicz interpolation theorem and the Calderon-Zygmund theorem. We obtain the mean oscillation (or BMO estimate) of solutions and then apply the Marcinkiewicz interpolation theorem to obtain L_p -estimates of solutions. To go from L_p -theory to $L_q(L_p)$ -theory we show that the kernel appeared in the representation of solutions for equations with constant coefficients satisfies the conditions needed for the Calderon-Zygmund theorem. Perturbation and fixed point arguments are used to handle the variable coefficients and the nonlinear term respectively.

The article is organized as follows. Some properties of the fractional derivatives and our main result, Theorem 2.9, are presented in Section 2. The representation of

solutions to a model equation and an L_2 -estimate of solutions are given in Section 3, and BMO and an $L_q(L_p)$ -estimate of solutions to a model equation are obtained in Section 4. The proof of Theorem 2.9 is given in Section 5, and sharp estimates of kernels related to the representation of solutions are obtained in Section 6.

We finish the introduction with some notation used in this article. As usual $\mathbb{N} = \{1, 2, \dots\}$, \mathbf{R}^d stands for the Euclidean space of points $x = (x^1, \dots, x^d)$, $B_r(x) := \{y \in \mathbf{R}^d : |x - y| < r\}$, and $B_r := B_r(0)$. For multi-indices $\gamma = (\gamma_1, \dots, \gamma_d)$, $\gamma_i \in \{0, 1, 2, \dots\}$, $x \in \mathbf{R}^d$, and functions $u(x)$ we set

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D_x^\gamma u = D_1^{\gamma_1} \cdot \dots \cdot D_d^{\gamma_d} u,$$

$$x^\gamma = (x^1)^{\gamma_1} (x^2)^{\gamma_2} \dots (x^d)^{\gamma_d}, \quad |\gamma| = \gamma_1 + \dots + \gamma_d.$$

We also use D_x^m to denote a partial derivative of order m with respect to x . For an open set $\Omega \subset \mathbf{R}^d$ by $C_c^\infty(\Omega)$ we denote the set of infinitely differentiable functions with compact support in Ω . For a Banach space F and $p > 1$ by $L_p(U, F)$ we denote the set of F -valued Lebesgue-measurable functions u on Ω satisfying

$$\|u\|_{L_p(\Omega, F)} = \left(\int_\Omega \|u(x)\|_F^p dx \right)^{1/p} < \infty.$$

We write $f \in L_{p,loc}(U, F)$ if $\zeta f \in L_p(U, F)$ for any real-valued $\zeta \in C_c^\infty(U)$. Also $L_p(\Omega) = L_p(\Omega, \mathbf{R})$ and $L_p = L_p(\mathbf{R}^d)$. We use “:=” to denote a definition. By \mathcal{F} and \mathcal{F}^{-1} we denote the d -dimensional Fourier transform and the inverse Fourier transform respectively, i.e.

$$\mathcal{F}(f)(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{-ix \cdot \xi} f(x) dx, \quad \mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{ix \cdot \xi} f(\xi) d\xi.$$

For a Lebesgue set $A \subset \mathbf{R}^d$, we use $|A|$ to denote its Lebesgue measure and by $1_A(x)$ we denote the indicator of A . For a complex number z , $\Re[z]$ is the real part of z . Finally if we write $N = N(a, b, \dots)$, this means that the constant N depends only on a, b, \dots .

2. MAIN RESULTS

We fix $T \in (0, \infty)$ throughout the article. For $\alpha > 0$ denote

$$k_\alpha(t) := t^{\alpha-1} \Gamma(\alpha)^{-1}, \quad t > 0,$$

where $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t} dt$. For functions $\varphi \in L_1((0, T))$ the Riemann-Liouville fractional integral of the order $\alpha > 0$ is defined as

$$I^\alpha \varphi(t) = k_\alpha * \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds.$$

It is easy to check that

$$I^{\alpha+\beta} \varphi(t) = I^\alpha I^\beta \varphi(t) \quad \forall \alpha, \beta > 0.$$

Also, by Jensen’s inequality, for any $p \in [1, \infty]$,

$$\|I^\alpha \varphi\|_{L_p((0, T))} \leq N(T, \alpha) \|\varphi\|_{L_p((0, T))}. \quad (2.1)$$

It is also known (see e.g. [39]) that $I^\alpha : B^\lambda \rightarrow C^{\lambda+\alpha}$ is a bounded operator if $\lambda \geq 0$ and $\lambda + \alpha < 1$, where $B^0 = L_\infty$ and $B^\lambda = C^\lambda$ if $\lambda > 0$.

Let $k \in \mathbb{N}$, $k-1 \leq \alpha < k$, and $f^{(k-1)}(t)$ be absolutely continuous, where $f^{(k-1)}(t)$ denotes the $(k-1)$ -th derivative of function f . Then the Caputo fractional derivative of order $\alpha > 0$ is defined as

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-s)^{k-\alpha-1} f^{(n)}(s) ds \quad (2.2)$$

$$= \frac{1}{\Gamma(k-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{k-\alpha-1} \left[f^{(k-1)}(s) - f^{(k-1)}(0) \right] ds. \quad (2.3)$$

Note that (2.3) (“Kochubei extension”) is defined for a broader class of functions.

Let $q \geq 1$. For functions $f \in C^k([0, T])$, we denote

$$\|f\|_{H_q^\alpha(T)} = \left(\int_0^T |f|^q dt \right)^{1/q} + \left(\int_0^T |\partial_t^\alpha f|^q dt \right)^{1/q}.$$

The following lemma shows that it is irreverent whether one uses (2.2) or (2.3) as the definition of ∂_t^α for functions in $H_q^\alpha(T)$.

Lemma 2.1. *The closures $H_q^\alpha(T)$ and $\tilde{H}_q^\alpha(T)$ of $C^k([0, T])$ in the space $L_q((0, T))$ with respect to norms $\|\cdot\|_{H_q^\alpha}$ related to (2.2) and (2.3) respectively coincide.*

Proof. This is obvious because (2.2) and (2.3) are equal for functions $f \in C^k([0, T])$. \square

Next we introduce another fractional derivative. Let D_t^α denote the Riemann-Liouville fractional derivative of order α which is defined as

$$D_t^\alpha \varphi(t) = \frac{d}{dt} (I^{1-\alpha} \varphi)(t), \quad \alpha \in (0, 1). \quad (2.4)$$

It is obvious that

$$\partial_t^\alpha \varphi(t) = D_t^\alpha (\varphi - \varphi(0)) = D_t^\alpha \varphi(t) - \frac{\varphi(0)}{t^\alpha \Gamma(1-\alpha)}, \quad \alpha \in (0, 1). \quad (2.5)$$

It is easy to check for any $\varphi \in L_1((0, T))$,

$$D_t^\alpha I^\alpha \varphi = \varphi, \quad \alpha \in (0, 1). \quad (2.6)$$

Similarly, the equality

$$I^\alpha D_t^\alpha \varphi = \varphi, \quad \alpha \in (0, 1) \quad (2.7)$$

also holds if $I^{1-\alpha} \varphi$ is absolutely continuous and $I^{1-\alpha} \varphi(0) = 0$.

Definition 2.2. *Let $k-1 \leq \alpha < k$ and $f \in H_q^\alpha(T)$. We write $f(0) = 0$ if there exists a sequence $f_n \in C^k([0, T])$ such that $f_n(0) = 0$ and $f_n \rightarrow f$ in $H_q^\alpha(T)$. Similarly if $\alpha > 1$ we write $f'(0) = 0$ if $f'_n(0) = 0$ for all n .*

The following lemma gives sufficient and necessary conditions for $f \in H_q^\alpha(T)$ and $f(0) = 0$ (or $f'(0) = 0$).

Lemma 2.3. (i) *Let $\alpha \in (0, 1)$ and $q > 1$. Then $f \in H_q^\alpha(T)$ and $f(0) = 0$ if and only if $f \in L_q((0, T))$, $I^{1-\alpha} f \in H_q^1(T)$, $I^{1-\alpha} f(t)$ is continuous, and $I^{1-\alpha} f(0) = 0$.*

(ii) *Let $\alpha \in (1, 2)$ and $q > 1$. Then $f \in H_q^\alpha(T)$ and $f'(0) = 0$ if and only if $f \in H_q^1(T)$, $I^{2-\alpha} f' \in H_q^1(T)$, $I^{2-\alpha} f'(t)$ is continuous, and $I^{2-\alpha} f'(0) = 0$.*

Proof. (i) Suppose $f \in H_q^\alpha(T)$ and $f(0) = 0$. Take a sequence $f_n \in C^1([0, T])$ satisfying $f_n(0) = 0$ and $f_n \rightarrow f$ in $H_q^\alpha(T)$. Then we have $f_n \rightarrow f$ and $\frac{d}{dt}(I^{1-\alpha}f_n) \rightarrow \partial_t^\alpha f$ in $L_q((0, T))$. It follows that $I^{1-\alpha}f_n$ converges to $I^{1-\alpha}f$ in the space $H_q^1((0, T))$. It also follows that

$$I^{1-\alpha}f(t) = \int_0^t \partial_t^\alpha f(s) ds, \quad t \leq T \quad (a.e.).$$

Since $1 - 1/q > 0$, by the Sobolev embedding theorem $I^{1-\alpha}f(t)$ is continuous in t and thus the above equality holds for all t , which implies $I^{1-\alpha}f(0) = 0$.

Next we prove the “if” part of (i). By the assumption, we can choose a function $g \in L_q((0, T))$ so that

$$I^{1-\alpha}f(t) = \int_0^t g(s) ds, \quad \forall t \leq T.$$

Take a sequence of functions $g_n \in C^1([0, T])$ which converges to g in $L_q((0, T))$. Define $f_n = I^\alpha g_n$. Then $f_n \in C^1([0, T])$, $f_n(0) = 0$, and

$$f_n \rightarrow I^\alpha g = f \quad \text{in } L_q((0, T)).$$

Also

$$\partial_t^\alpha f_n = \partial_t^\alpha I^\alpha g_n = g_n \rightarrow \partial_t^\alpha f \quad \text{in } L_q((0, T)).$$

(ii) The proof is very similar to (i). We only explain how one can choose a sequence to prove the “if” part.

By the assumption $f \in H_q^1(T)$ and $q > 1$, we may assume $f \in C([0, T])$. Take $g \in L_q((0, T))$ so that $I^{2-\alpha}f' = \int_0^t g(s) ds$. We choose $g_n \in C^2([0, T])$ which converges to g in $L_q((0, T))$. Define $f_n(t) = I^\alpha g_n(t) + f(0)$. Then $f_n'(0) = I^{\alpha-1}g_n(0) = 0$, $f_n \rightarrow f$ in $L_q((0, T))$, and $\partial_t^\alpha f_n$ is a Cauchy sequence in $L_q((0, T))$. Thus $f \in H_p^\alpha(T)$ and $f'(0) = 0$.

The lemma is proved. \square

Next we introduce our solution space $\mathbb{H}_{q,p}^{\alpha,k}(T)$ and related notation. Roughly speaking, we write $u \in \mathbb{H}_{q,p}^{\alpha,k}(T)$ if and only if

$$u, \partial_t^\alpha u, D_x^k u \in L_q((0, T), L_p).$$

For $p, q > 1$ and $k = 0, 1, 2, \dots$, we denote

$$H_p^k = H_p^k(\mathbf{R}^d) = \{u \in L_p(\mathbf{R}^d) : D_x^\gamma u \in L_p(\mathbf{R}^d), |\gamma| \leq k\},$$

$$\mathbb{H}_{q,p}^{0,k}(T) = L_q((0, T), H_p^k), \quad \mathbb{L}_{q,p}(T) := \mathbb{H}_{q,p}^{0,0}(T),$$

where D_x^γ are derivatives in the distributional sense. Thus $u \in \mathbb{H}_{q,p}^{0,k}(T)$ if and only if $u(t, \cdot)$ is H_p^k -valued measurable function satisfying

$$\|u\|_{\mathbb{H}_{q,p}^{0,k}(T)} := \left[\int_0^T \|u\|_{H_p^k}^q ds \right]^{1/q} < \infty.$$

We extend the real-valued time fractional Sobolev space to $L_p(\mathbf{R}^d)$ -valued one. In other words, we consider the completion of $C^2([0, T] \times \mathbf{R}^d) \cap \mathbb{L}_{q,p}(T)$ with respect to norm

$$\|\cdot\|_{\mathbb{L}_{q,p}(T)} + \|\partial_t^\alpha \cdot\|_{\mathbb{L}_{q,p}(T)}$$

in $\mathbb{L}_{q,p}(T)$.

Definition 2.4. For $\alpha \in (0, 2)$ we say $u \in \mathbb{H}_{q,p}^{\alpha,0}(T)$ if and only if there exists a sequence $u_n \in C^2([0, T] \times \mathbf{R}^d) \cap \mathbb{L}_{q,p}(T)$ so that $\sup_n \|\partial_t^\alpha u_n\|_{\mathbb{L}_{q,p}(T)} < \infty$, and

$$\|u - u_n\|_{\mathbb{L}_{q,p}(T)} \rightarrow 0 \quad \text{and} \quad \|\partial_t^\alpha u_n - \partial_t^\alpha u_m\|_{\mathbb{L}_{q,p}(T)} \rightarrow 0$$

as n and m go to infinity. We call this sequence u_n a defining sequence of u . For $u \in \mathbb{H}_{q,p}^{\alpha,0}(T)$, we define

$$\partial_t^\alpha u = \lim_{n \rightarrow \infty} \partial_t^\alpha u_n \quad \text{in} \quad \mathbb{L}_{q,p}(T),$$

where u_n is a defining sequence of u . Obviously $\mathbb{H}_{q,p}^{\alpha,0}(T)$ is a Banach space with the norm

$$\|u\|_{\mathbb{H}_{q,p}^{\alpha,0}(T)} = \|u\|_{\mathbb{L}_{q,p}(T)} + \|\partial_t^\alpha u\|_{\mathbb{L}_{q,p}(T)}.$$

Definition 2.5. For $u \in \mathbb{H}_{q,p}^{\alpha,0}(T)$, we say that $u(0, x) = 0$ if and only if there exists a defining sequence u_n such that

$$u_n(0, x) = 0 \quad \forall x \in \mathbf{R}^d, \quad \forall n \in \mathbb{N}.$$

Similarly we say that $u(0, \cdot) = 0$ and $\frac{\partial}{\partial t} u(0, \cdot) = 0$ if and only if there exists a defining sequence u_n such that

$$u_n(0, x) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} u_n(0, x) = 0 \quad \forall x \in \mathbf{R}^d, \quad \forall n \in \mathbb{N}.$$

Let

$$\mathbb{H}_{q,p}^{\alpha,k}(T) := \mathbb{H}_{q,p}^{\alpha,0}(T) \cap \mathbb{H}_{q,p}^{0,k}(T)$$

and $\mathbb{H}_{q,p,0}^{\alpha,k}(T)$ be the subspace of $\mathbb{H}_{q,p}^{\alpha,k}(T)$ such that

$$u(0, \cdot) = 0 \quad \text{if} \quad \alpha \in (0, 1]$$

and

$$u(0, \cdot) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} u(0, \cdot) = 0 \quad \text{if} \quad \alpha \in (1, 2).$$

Theorem 2.6. (i) The space $\mathbb{H}_{q,p}^{\alpha,k}(T)$ is a Banach space with the norm

$$\|u\|_{\mathbb{H}_{q,p}^{\alpha,k}(T)} := \|u\|_{\mathbb{H}_{q,p}^{0,k}(T)} + \|u\|_{\mathbb{H}_{q,p}^{\alpha,0}(T)}.$$

(ii) The space $\mathbb{H}_{q,p,0}^{\alpha,k}(T)$ is a closed subspace of $\mathbb{H}_{q,p}^{\alpha,k}(T)$.

(iii) $C_c^\infty(\mathbf{R}_+^{d+1})$ is dense in $\mathbb{H}_{q,p,0}^{\alpha,k}(T)$.

(iv) For any $u \in \mathbb{H}_{q,p,0}^{\alpha,2}(T)$,

$$\|u(t)\|_{L_p} \leq N(\alpha) \int_0^t (t-s)^{\alpha-1} \|\partial_t^\alpha u(s)\|_{L_p} ds, \quad t \leq T \quad (a.e.). \quad (2.8)$$

Consequently,

$$\|u\|_{\mathbb{L}_{q,p}(t)}^q \leq N(q, \alpha, T) \int_0^t \int_0^s (s-r)^{\alpha-1} \|\partial_t^\alpha u(r)\|_{L_p}^q dr ds, \quad \forall t \leq T.$$

Proof. (i) This is obvious because both $\mathbb{H}_{q,p}^{\alpha,0}(T)$ and $\mathbb{H}_{q,p}^{0,k}(T)$ are Banach spaces.

(ii) Suppose $u_n \in \mathbb{H}_{q,p,0}^{\alpha,k}(T)$ and $u \in \mathbb{H}_{q,p}^{\alpha,k}(T)$ so that $u_n \rightarrow u$ in $\mathbb{H}_{q,p}^{\alpha,k}(T)$. Since $u_n \in \mathbb{H}_{q,p,0}^{\alpha,k}(T)$ for each n , we can find a $v_n \in C^2([0, T] \times \mathbf{R}^d) \cap \mathbb{L}_{q,p}(T)$ so that

$$\begin{aligned} \|u_n - v_n\|_{\mathbb{L}_{q,p}(T)} &< \frac{1}{n} \quad \text{and} \quad \|\partial_t^\alpha u_n - \partial_t^\alpha v_n\|_{\mathbb{L}_{q,p}(T)} < \frac{1}{n}, \\ v_n(0, x) &= 0 \quad \forall x \in \mathbf{R}^d, \quad \text{if} \quad \alpha \in (0, 1], \end{aligned}$$

and

$$v_n(0, x) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} v_n(0, x) = 0 \quad \forall x \in \mathbf{R}^d, \quad \text{if } \alpha \in (1, 2).$$

Therefore $u \in \mathbb{H}_{q,p,0}^{\alpha,k}(T)$ because obviously

$$\|v_n - u\|_{\mathbb{H}_{q,p}^{\alpha,0}(T)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This certainly proves (ii).

(iii) We take nonnegative smooth functions $\eta_1 \in C_c^\infty((1, 2))$, $\eta_2 \in C_c^\infty(\mathbf{R}^d)$, and $\eta_3 \in C_c^\infty(\mathbf{R}^d)$ so that

$$\int_0^\infty \eta_1(t) dt = 1, \quad \int_{\mathbf{R}^d} \eta_2(x) dx = 1, \quad \text{and} \quad \eta_3(x) = 1 \quad \text{if } |x| \leq 1.$$

For $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, we define

$$\begin{aligned} \eta_{1,\varepsilon_1}(t) &= \varepsilon_1^{-1} \eta_1(t/\varepsilon_1), \quad \eta_{2,\varepsilon}(x) = \varepsilon_2^{-d} \eta_2(x/\varepsilon_2), \\ u^{\varepsilon_1}(t, x) &= \int_0^\infty u(s, x) \eta_{1,\varepsilon_1}(t-s) ds, \\ u^{\varepsilon_1, \varepsilon_2}(t, x) &= \int_{\mathbf{R}^d} \int_0^\infty u(s, y) \eta_{1,\varepsilon_1}(t-s) \eta_{2,\varepsilon_2}(x-y) ds dy, \end{aligned}$$

and

$$u^{\varepsilon_1, \varepsilon_2, \varepsilon_3}(t, x) = \eta(t) \eta_3(\varepsilon_3 x) \int_{\mathbf{R}^d} \int_0^\infty u(s, y) \eta_{1,\varepsilon_1}(t-s) \eta_{2,\varepsilon_2}(x-y) ds dy,$$

where $\eta \in C^\infty([0, \infty))$ such that $\eta(t) = 1$ for all $t \leq T$ and vanishes for all large t . Due to the condition $\eta_1 \in C_c^\infty((1, 2))$, it holds that

$$u^{\varepsilon_1, \varepsilon_2, \varepsilon_3}(t, x) = 0 \quad \forall t < \varepsilon_1, \quad \forall x \in \mathbf{R}^d.$$

We can easily check that for any $u \in \mathbb{H}_{q,p,0}^{\alpha,k}(T)$

$$\partial_t^\alpha u^{\varepsilon_1}(t) = (\partial_t^\alpha u)^{\varepsilon_1}(t).$$

Hence for any given $\varepsilon > 0$ we have

$$\begin{aligned} &\|u - u^{\varepsilon_1, \varepsilon_2, \varepsilon_3}\|_{\mathbb{H}_{q,p}^{\alpha,k}(T)} \\ &\leq \|u - u^{\varepsilon_1}\|_{\mathbb{H}_{q,p}^{\alpha,k}(T)} + \|u^{\varepsilon_1} - u^{\varepsilon_1, \varepsilon_2}\|_{\mathbb{H}_{q,p}^{\alpha,k}(T)} + \|u^{\varepsilon_1, \varepsilon_2} - u^{\varepsilon_1, \varepsilon_2, \varepsilon_3}\|_{\mathbb{H}_{q,p}^{\alpha,k}(T)} \leq \varepsilon \end{aligned}$$

if $\varepsilon_1, \varepsilon_2$, and ε_3 are small enough. Therefore (iii) is proved.

(iv) Due to (iii), it is enough to prove (2.8) only for $u \in C_c^\infty(\mathbf{R}^{d+1})$. Denote $f := \partial_t^\alpha u$. One can easily check

$$u(t) = \int_0^t k_\alpha(t-s) f(s) ds, \quad \forall t \leq T$$

in the space L_p , which clearly implies (2.8) due to the generalized Minkowski inequality. The theorem is proved. \square

Assumption 2.7. Let $f(u) = f(t, x, u)$ and $f_0 = f(t, x, 0)$.

(i) There exist $0 = T_0 < T_1 < \dots < T_\ell = T$ and functions $a_k^{ij}(t, x)$ such that

$$a^{ij}(t, x) = \sum_{k=1}^{\ell} a_k^{ij}(t, x) I_{(T_{k-1}, T_k]}(t), \quad (a.e.).$$

(ii) There exist constants $\delta, K > 0$ so that for any k, t , and x

$$\delta|\xi|^2 \leq a_k^{ij}(t, x)\xi^i\xi^j \leq K|\xi|^2, \quad \forall \xi \in \mathbf{R}^d, \quad (2.9)$$

and

$$|a_k^{ij}(t, x)| + |b^i(t, x)| + |c(t, x)| \leq K.$$

(iii) The coefficients a_k^{ij} are uniformly continuous on $(t_{k-1}, t_k] \times \mathbf{R}^d$ for all $k = 1, \dots, \ell$ and $i, j = 1, \dots, d$.

(iv) $f_0 \in \mathbb{L}_{q,p}(T)$ and $f(u)$ satisfies the following continuity property: for any $\varepsilon > 0$, there exists a constant $K_\varepsilon > 0$ such that

$$\|f(t, x, u) - f(t, x, v)\|_{L_p} \leq \varepsilon \|u - v\|_{H_p^2} + K_\varepsilon \|u - v\|_{L_p}, \quad (2.10)$$

for any (t, x) and $u, v \in H_p^2$.

If $p \neq q$ then we need an additional condition (see the comment below (5.4) for the reason).

Assumption 2.8. If $p \neq q$ then $\lim_{|x| \rightarrow \infty} a^{ij}(t, x)$ exists uniformly in $t \in (0, T)$.

Here is the main result of this article. The proof will be given in Section 5.

Theorem 2.9. Let $p, q > 1$ and Assumptions 2.7 and 2.8 hold. Then the equation

$$\partial_t^\alpha u = a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f(u), \quad t > 0 \quad (2.11)$$

admits a unique solution u in the class $\mathbb{H}_{q,p,0}^{\alpha,2}(T)$, and for this solution it holds that

$$\|u\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)} \leq N_0 \|f_0\|_{\mathbb{L}_{q,p}(T)}, \quad (2.12)$$

where N_0 depends only on $d, p, q, \delta, K, K_\varepsilon, T, \ell$, and the modulus of continuity of a_k^{ij} .

Remark 2.10. (i) Due to the definition of our solution space $\mathbb{H}_{q,p,0}^{\alpha,2}(T)$, the zero initial condition is given to equation (2.11), that is

$$u(0, x) = 0 \quad \text{and additionally} \quad \frac{\partial u}{\partial t}(0, x) = 0 \quad \text{if } \alpha > 1.$$

(ii) Some examples of $f(u)$ satisfying (2.10) can be found e.g. in [19]. For instance, let $\kappa := 2 - d/p > 0$, $h = h(x) \in L_p$, and

$$f(x, u) := h(x) \sup_x |u|.$$

Then by the Sobolev embedding theorem,

$$\|f(u) - f(v)\|_{L_p} \leq \|h\|_{L_p} \sup_x |u - v| \leq N \|u - v\|_{H_p^\kappa} \leq \varepsilon \|u - v\|_{H_p^2} + K \|u - v\|_{L_p}.$$

Similarly one can show that $f(t, x, u) := a(t, x)(-\Delta)^\delta u$ also satisfies (2.10) if $a(t, x)$ is bounded and $\delta \in (0, 1)$.

3. SOME PRELIMINARIES

In this section we introduce some estimates of kernels related to the representation of a solution, and we also present an L_2 -estimate of a solution.

The Mittag-Leffler function $E_\alpha(z)$ is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}, \alpha > 0.$$

The series converges for any $z \in \mathbb{C}$, and $E_\alpha(z)$ is an entire function. Using

$$\partial_t^\alpha t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \alpha)} t^{\beta - \alpha}, \quad \beta \geq \alpha$$

one can easily check that for any constant λ ,

$$\varphi(t) := E_\alpha(\lambda t^\alpha)$$

satisfies $\varphi(0) = 1$ (also $\varphi'(0) = 0$ if $\alpha > 1$) and

$$\partial_t^\alpha \varphi = \lambda \varphi, \quad t > 0.$$

By taking the Fourier transform to the equation

$$\partial_t^\alpha u = \Delta u, \quad t > 0, \quad u(0) = h, \quad (\text{and } u'(0) = 0 \text{ if } \alpha > 1)$$

one can formally get $\mathcal{F}(u)(t, \xi) = E_\alpha(-t^\alpha |\xi|^2) \tilde{h}$. Thus it is naturally needed to find an integrable function $p(t, x)$ satisfying $\mathcal{F}(p(t, \cdot))(\xi) = E_\alpha(-t^\alpha |\xi|^2)$. It is known that (see e.g. (12) in [10, Theorem 1.3-4])

$$E_\alpha(-t) \sim \frac{1}{1 + |t|}, \quad t > 0. \quad (3.1)$$

Therefore, $E_\alpha(-t^\alpha |\xi|^2)$ is integrable only if $d = 1$. If $d \geq 2$ then $\mathcal{F}^{-1}(E_\alpha(-t^\alpha |\xi|^2))$ might be understood as an improper integral. However, in this article we do not consider $\mathcal{F}^{-1}(E_\alpha(-t^\alpha |\xi|^2))$.

Lemma 3.1. (i) Let $d \geq 1$ and $\alpha \in (0, 2)$. Then there exists a function $p(t, x)$ such that $p(t, \cdot)$ is integrable in \mathbf{R}^d and

$$\mathcal{F}(p(t, \cdot))(\xi) = E_\alpha(-t^\alpha |\xi|^2).$$

(ii) Let $m, n = 0, 1, 2, \dots$ and denote $R = t^{-\alpha} |x|^2$. Then there exist constants C and σ depending only on m, n, d , and α so that if $R \geq 1$

$$|\partial_t^n D_x^m p(t, x)| \leq N t^{\frac{-\alpha(d+m)}{2} - n} \exp\{-\sigma t^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{2}{2-\alpha}}\}, \quad (3.2)$$

and if $R \leq 1$

$$|\partial_t^n D_x^m p(t, x)| \leq N |x|^{-d-m} t^{-n} \left(R + R \ln R \cdot 1_{d=2, m=0} + R^{1/2} \cdot 1_{d=1, m=0} \right). \quad (3.3)$$

By (3.2), $p(t, x)$ is absolutely continuous on $(0, T)$ and $\lim_{t \rightarrow 0} p(t, x) = 0$ if $x \neq 0$. Thus we can define

$$q(t, x) := \begin{cases} I^{\alpha-1} p(t, x), & \alpha \in (1, 2) \\ D_t^{1-\alpha} p(t, x), & \alpha \in (0, 1). \end{cases}$$

Since $p(0, x) = 0$ for $x \neq 0$, $D_t^{1-\alpha} p(t, x) = \partial_t^{1-\alpha} p(t, x)$.

Lemma 3.2. (i) Let $d \geq 1$, $\alpha \in (0, 2)$, and $m, n = 0, 1, 2, \dots$. Denote $R = t^{-\alpha}|x|^2$. Then there exist constants N and σ depending only on m, n, d , and α so that if $R \geq 1$

$$|\partial_t^n D_x^m q(t, x)| \leq N t^{\frac{-\alpha(d+m)}{2} - n + \alpha - 1} \exp\{-\sigma t^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{2}{2-\alpha}}\}, \quad (3.4)$$

and if $R \leq 1$

$$\begin{aligned} |\partial_t^n D_x^m q(t, x)| &\leq N |x|^{-d-m} t^{-n+\alpha-1} (R^2 + R^2 \ln R \cdot 1_{d=2}) \\ &\quad + N |x|^{-d} t^{-n+\alpha-1} \left(R^{1/2} \cdot 1_{d=1} + R \cdot 1_{d=2} + R^2 \ln R \cdot 1_{d=4} \right) 1_{m=0}. \end{aligned} \quad (3.5)$$

(ii) For any $t \neq 0$ and $x \neq 0$,

$$\partial_t^\alpha p = \Delta p, \quad \frac{\partial p}{\partial t} = \Delta q. \quad (3.6)$$

One can find similar statements of Lemmas 3.1 and 3.2 in [11, 12, 18, 35]. For the sake of completeness, we give an independent and rigorous proof in Section 6.

Corollary 3.3. Let $0 < \varepsilon < T$. Then

$$\int_{\mathbf{R}^d} \sup_{t \in [\varepsilon, T]} |D_x^m q(t, x)| dx < \infty, \quad m = 0, 1, 2.$$

and

$$\int_{\mathbf{R}^d} \sup_{t \in [\varepsilon, T]} |D_t^m p(t, x)| dx < \infty, \quad m = 0, 1.$$

Proof. By (3.4) and (3.5), it follows that for large $|x|$

$$\sup_{t \in [\varepsilon, T]} |D_x^m q(t, x)| \leq N e^{-c|x|^{\frac{2}{2-\alpha}}}$$

and for small $|x|$

$$\sup_{t \in [\varepsilon, T]} |D_x^m q(t, x)| \leq N |x|^{-d-m} (|x|^3 + |x| \cdot 1_{m=0}),$$

where N and c depend only on d, α, T , and ε . This certainly proves the assertion related to q , and p is handled similarly. The corollary is proved. \square

Lemma 3.4. Let $\lambda > 0$, $\alpha \in (0, 2)$, and ϕ be a continuous function on $[0, \infty)$ so that the Laplace transforms of ϕ and $\partial_t^\alpha \phi$ exist,

$$\partial_t^\alpha \phi + \lambda \phi = f(t), \quad t > 0, \quad (3.7)$$

and $\phi(0) = 0$ (additionally $\phi'(0) = 0$ if $\alpha \in (1, 2)$). Moreover we assume for each $s > 0$,

$$e^{-st} \int_0^t \partial_t^\alpha \phi(r) dr \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Then

$$\phi(t) = \int_0^t H_{\alpha, \lambda}(t-s) f(s) ds, \quad (3.8)$$

where $H_{\alpha, \lambda}(t) = D_t^{1-\alpha} E_\alpha(-\lambda t^\alpha)$ if $\alpha \in (0, 1)$ and $H_{\alpha, \lambda}(t) = I^{\alpha-1} E_\alpha(-\lambda t^\alpha)$ otherwise.

Proof. First, recall that $\varphi(t) := E_\alpha(-t^\alpha)$ satisfies $\varphi(0) = 1$ and $\partial_t^\alpha \varphi = -\varphi$. Let \mathcal{L} denote the Laplace transform. Then from $\mathcal{L}[\partial_t^\alpha \varphi] = -\mathcal{L}[\varphi]$ we get

$$\mathcal{L}[\varphi](s) := \int_0^\infty e^{-st} \varphi(t) dt = \frac{s^{\alpha-1}}{s^\alpha + 1}.$$

Here, we used the following facts: for $\beta \in (0, 1)$,

$$\mathcal{L}[h'] = s\mathcal{L}[h](s), \quad \mathcal{L}[h * g](s) = \mathcal{L}[h](s) \cdot \mathcal{L}[g](s), \quad \mathcal{L}[k_{1-\beta}](s) = s^{\beta-1}. \quad (3.9)$$

It follows that

$$\mathcal{L}[E_\alpha(-\lambda t^\alpha)](s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda}, \quad \mathcal{L}[H_{\alpha,\lambda}](s) = \frac{1}{s^\alpha + \lambda}. \quad (3.10)$$

On the other hand, taking the Laplace transform to (3.7) and using (3.9) we get

$$\mathcal{L}[\phi](s) = \frac{1}{s^\alpha + \lambda} \cdot \mathcal{L}[f](s), \quad s > 0.$$

This and (3.10) certainly prove the lemma, because to prove equality (3.8) it is enough to show that two functions under consideration have the same Laplace transform. \square

Lemma 3.5. (i) Let $u \in C_c^\infty(\mathbf{R}_+^{d+1})$ and denote $f := \partial_t^\alpha u - \Delta u$. Then

$$u(t, x) = \int_0^t \int_{\mathbf{R}^d} q(t-s, x-y) f(s, y) dy ds. \quad (3.11)$$

(ii) Let $f \in C_c^\infty(\mathbf{R}_+^{d+1})$ and define u as in (3.11). Then u satisfies $\partial_t^\alpha u = \Delta u + f$.

Proof. (i) Since q is integrable on $(0, T) \times \mathbf{R}^d$ (see Lemma 3.2), it is enough to prove

$$\hat{u}(t, \xi) = \int_0^t \hat{q}(t-s, \xi) \hat{f}(s, \xi) ds, \quad (3.12)$$

where \hat{f} denotes the Fourier transform of f with respect to x , i.e.

$$\hat{f}(s, \xi) := \mathcal{F}(f(s, \cdot))(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{-ix \cdot \xi} f(s, x) dx.$$

First note that from the definition of f we get

$$\hat{f}(t, \xi) = \partial_t^\alpha \hat{u}(t, \xi) + |\xi|^2 \hat{u}(t, \xi), \quad \forall t > 0.$$

Therefore by Lemma 3.4

$$\hat{u}(t, \xi) = \int_0^t H_{\alpha, |\xi|^2}(t-s) \hat{f}(s, \xi) ds.$$

Hence it is enough to prove

$$\hat{q}(t, \xi) = H_{\alpha, |\xi|^2}(t). \quad (3.13)$$

Denote $c_d = (2\pi)^{-d/2}$. If $\alpha \in (0, 1)$ then by the definition

$$\hat{q}(t, x) = c_d \int_{\mathbf{R}^d} e^{-ix \cdot \xi} q(t, x) dx = c_d \int_{\mathbf{R}^d} e^{-ix \cdot \xi} \left[\frac{d}{dt} \int_0^t k_\alpha(t-s) p(s, x) ds \right] dx.$$

Since $q(t, \cdot)$ is integrable in \mathbf{R}^d uniformly in a neighborhood of $t > 0$ (see Corollary 3.3), one can take the derivative $\frac{d}{dt}$ out of the integral. After this using Fubini's theorem we get

$$\hat{q} = \frac{d}{dt} \int_0^t k_\alpha(t-s) \left[c_d \int_{\mathbf{R}^d} e^{-ix \cdot \xi} p(s, x) dx \right] ds = D_t^{1-\alpha} E_\alpha(-t^\alpha |\xi|^2).$$

Hence (3.13) and (i) are proved. The case $\alpha \in [1, 2)$ is easier and we skip the proof.

(ii) Taking the Fourier transform to (3.11) we get

$$\hat{u}(t, \xi) = \int_0^t H_{\alpha, |\xi|^2}(t-s) \hat{f}(s, \xi) ds.$$

Note that if one defines ϕ as in (3.8) then it satisfies (3.7). Consequently, \hat{u} satisfies

$$\partial_t^\alpha \hat{u}(t, \xi) + |\xi|^2 \hat{u}(t, \xi) = \hat{f}(t, \xi),$$

and this certainly proves the equality $\partial_t^\alpha u = \Delta u + f$, because $\hat{f}(t, \cdot) = 0$ if t is small enough and thus for each $t > 0$,

$$\mathcal{F}(\partial_t^\alpha u(t, \cdot))(x) = \int_0^t H_{\alpha, |\xi|^2}(t-s) \partial_s^\alpha \hat{f}(s, \xi) ds = \partial_t^\alpha \hat{u}(t, \xi).$$

The lemma is proved. \square

Now we define the operator \mathcal{G} by

$$\mathcal{G}f(t, x) = \int_{-\infty}^t \int_{\mathbf{R}^d} q(t-s, x-y) \Delta f(s, y) dy ds.$$

Since q is integrable on $(0, T) \times \mathbf{R}^d$, $\mathcal{G}f$ is well-defined if $f \in C_c^\infty(\mathbf{R}^{d+1})$. Also recall that $D_x q(t, \cdot)$ and $D_x^2 q(t, \cdot)$ are integrable in \mathbf{R}^d for each $t > 0$, and therefore it follows that

$$\mathcal{G}f(t, x) = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{t-\varepsilon} \left[\int_{\mathbf{R}^d} \Delta q(t-s, x-y) f(s, y) dy \right] ds. \quad (3.14)$$

Lemma 3.6. *Let $f \in C_c^\infty(\mathbf{R}^{d+1})$. Then*

$$\|\mathcal{G}f\|_{L_2(\mathbf{R}^{d+1})} \leq N \|f\|_{L_2(\mathbf{R}^{d+1})} \quad (3.15)$$

where $N = N(\alpha, d)$.

Proof. Denote $q_M = q 1_{0 < t < M}$. One can easily check that q_M is integrable in \mathbf{R}^{d+1} . Denote $\mathcal{G}_M f = q_M * \Delta f$. Then by Parseval's identity

$$\|\mathcal{G}_M f\|_{L_2}^2 = \int_{\mathbf{R}^{d+1}} |\mathcal{F}_{d+1}(q_M * \Delta f)|^2 d\tau d\xi.$$

By the properties of the Fourier transform,

$$\mathcal{F}_{d+1}(q_M * \Delta f)(\tau, \xi) = -N(d) |\xi|^2 \mathcal{F}_{d+1}(q_M)(\tau, \xi) \mathcal{F}_{d+1}(f)(\tau, \xi), \quad \forall (\tau, \xi) \in \mathbf{R}^{d+1}.$$

Set

$$I_M(\tau, \xi) := -|\xi|^2 \mathcal{F}_{d+1}(q_M)(\tau, \xi) = \int_{0 < t < M} e^{-i\tau t} \mathcal{F}_d\left(\frac{\partial p}{\partial t}\right)(t, \xi) dt.$$

Now we claim that $I_M(\tau, \xi)$ is bounded uniformly for $M > 0$, i.e.

$$\sup_{M > 0, \tau, \xi} |I_M(\tau, \xi)| < \infty.$$

Then the claim implies

$$\|\mathcal{G}_M f\|_{L_2}^2 \leq N \|f\|_{L_2}^2, \quad (3.16)$$

where N is independent of M .

By the integration by parts and the change of variables,

$$I_M(\tau, \xi) = i \operatorname{sgn}(\tau) \int_0^{\tau M} e^{-\operatorname{sgn}(\tau) i t} E_\alpha(-(\frac{t}{|\tau|})^\alpha |\xi|^2) dt + e^{-i \tau M} E_\alpha(-M^\alpha |\xi|^2) - 1. \quad (3.17)$$

Denote for $\frac{\alpha\pi}{2} < \eta < \min\{\pi, \alpha\pi\}$

$$\Delta_\eta^* := \{z \in \mathbb{C} : |\pi - \operatorname{Arg} z| < \pi - \eta\}.$$

Then by [10, (1.3.12)] or [31, Theorem 1.6],

$$|E_\alpha(z)| = \frac{N}{1 + |z|}, \quad z \in \Delta_\eta^*.$$

Hence the function $E_\alpha(z)$ is bounded in Δ_η^* .

Let $\tau > 0$. We take a $\theta_0 \in (0, \pi/3)$ sufficiently small so that $-e^{-i\theta}, -e^{-i\alpha\theta} \in \Delta_\eta^*$ for any $\theta \in [0, \theta_0]$. Denote

$$C_{1,\tau M} = \{t : t \in [0, \tau M]\}, \quad C_{2,\tau M} = \{te^{-i\theta_0} : t \in [0, \tau M]\},$$

$$C_{3,\tau M} = \{\tau M e^{-i\theta} : \theta \in [0, \theta_0]\},$$

and define a contour $C_{\tau M} = C_{1,\tau M} \cup C_{2,\tau M} \cup C_{3,\tau M}$. Then the contour integral of the function $e^{-iz} E_\alpha(-(\frac{z}{\tau})^\alpha |\xi|^2)$ on $C_{\tau M}$ is zero for any $M > 0$. Since $E_\alpha(z)$ is bounded in Δ_η^* ,

$$\left| \int_{C_{3,\tau M}} e^{-iz} E_\alpha(-(\frac{z}{\tau})^\alpha |\xi|^2) dz \right| \leq \int_0^{\theta_0} (\tau M) e^{-\tau M \sin \theta} d\theta \leq N(\theta_0). \quad (3.18)$$

Also,

$$\left| \int_{C_{2,\tau M}} e^{-iz} E_\alpha(-(\frac{z}{\tau})^\alpha |\xi|^2) dz \right| \leq N \int_0^{\tau M} e^{-t \sin \theta_0} dt \leq N(\theta_0). \quad (3.19)$$

Note that θ_0 depends only on α . Hence by (3.17), (3.18), and (3.19), it follows that if $\tau > 0$ then $|I_M(\tau, \xi)|$ is bounded uniformly for M .

If $\tau < 0$ then we choose a contour $C'_{\tau M} = C_{1,\tau M} \cup C'_{2,\tau M} \cup C'_{3,\tau M}$ where

$$C'_{2,\tau M} = \{te^{i\theta_0} : t \in [0, \tau M]\}, \quad C'_{3,\tau M} = \{\tau M e^{i\theta} : \theta \in [0, \theta_0]\}.$$

Then the same arguments above go through. Thus our claim is proved.

To finish the proof, observe that for each $(t, x) \in \mathbb{R}^{d+1}$

$$q_M * \Delta f(t, x) = \mathcal{G}_M f(t, x) \rightarrow \mathcal{G} f(t, x) \quad \text{as } M \rightarrow \infty$$

since $\Delta f \in C_c^\infty(\mathbb{R}^{d+1})$. Therefore (3.16) and Fatou's lemma easily yields (3.15). \square

4. BMO AND $L_q(L_q)$ -ESTIMATE

In this section we obtain BMO and $L_q(L_p)$ -estimates for a solution of the model equation

$$\partial_t^\alpha u = \Delta u + f, \quad (t, x) \in (0, \infty) \times \mathbf{R}^d.$$

Recall that p is integrable with respect to x , $\mathcal{F}(p(t, \cdot)) = E_\alpha(-t^\alpha|\xi|^2)$, and q is defined as

$$q(t, x) = \begin{cases} I^{\alpha-1}p(t, x), & \alpha \in (1, 2) \\ D_t^{1-\alpha}p(t, x), & \alpha \in (0, 1). \end{cases}$$

Also recall that for $f \in C_c^\infty(\mathbf{R}_+^{d+1})$, it holds that

$$\mathcal{G}f = \int_{-\infty}^t \left[\int_{\mathbf{R}^d} \Delta q(t-s, x-y) f(s, y) dy \right] ds, \quad (4.1)$$

and by (3.15) the operator \mathcal{G} is continuously extended onto $L_2(\mathbf{R}^{d+1})$. We denote this extension by the same notation \mathcal{G} .

For a locally integrable function h on \mathbf{R}^{d+1} , we define the BMO semi-norm of h on \mathbf{R}^{d+1} as

$$\|h\|_{BMO(\mathbf{R}^{d+1})} = \sup_{Q \in \mathbb{Q}} \frac{1}{|Q|} \int_Q |h(t, x) - h_Q| dt dx$$

where $h_Q = \frac{1}{|Q|} \int_Q h(t, x) dt dx$ and

$$\mathbb{Q} := \{Q_\delta(t_0, x_0) = (t_0 - \delta^{2/\alpha}, t_0 + \delta^{2/\alpha}) \times B_\delta(x_0) : \delta > 0, (t_0, x_0) \in \mathbf{R}^{d+1}\}.$$

Denote $Q_\delta := Q_\delta(0, 0)$.

Lemma 4.1. *Let $f \in L_2(\mathbf{R}^{d+1})$ and vanish on $\mathbf{R}^{d+1} \setminus Q_{3\delta}$. Then*

$$\int_{Q_\delta} |\mathcal{G}f(t, x)| dt dx \leq N \|f\|_{L_\infty(\mathbf{R}^{d+1})},$$

where $N = N(d, \alpha)$.

Proof. By Hölder's inequality and Lemma 3.6,

$$\begin{aligned} \int_{Q_\delta} |\mathcal{G}f(t, x)| dt dx &\leq \left(\int_{Q_\delta} |\mathcal{G}f(t, x)|^2 dt dx \right)^{1/2} |Q_\delta|^{1/2} \\ &\leq \|\mathcal{G}f\|_{L_2(\mathbf{R}^{d+1})} |Q_\delta|^{1/2} \leq N \|f\|_{L_2(\mathbf{R}^{d+1})} |Q_\delta|^{1/2} \\ &= N \left(\int_{Q_{3\delta}} |f(t, x)|^2 dt dx \right)^{1/2} |Q_\delta|^{1/2} \leq N \|f\|_{L_\infty(\mathbf{R}^{d+1})} |Q_\delta|. \end{aligned}$$

The lemma is proved. \square

Denote

$$K(t, x) = 1_{t>0} \Delta q(t, x).$$

Due to Lemma 3.2(ii), we have $K(t, x) = 1_{t>0} \partial_t p(t, x)$. Furthermore the following scaling properties hold (see (6.21) and (6.12) for detail):

$$K(t, x) = 1_{t>0} t^{-1-\frac{\alpha d}{2}} K(1, t^{-\alpha/2} x) \quad (4.2)$$

$$\partial_t K(t, x) = 1_{t>0} t^{-2-\frac{\alpha d}{2}} (\partial_t K)(1, t^{-\alpha/2} x), \quad (4.3)$$

and

$$\frac{\partial}{\partial x^i} K(t, x) = 1_{t>0} t^{-1-\frac{\alpha(d+1)}{2}} \frac{\partial}{\partial x^i} K(1, t^{-\alpha/2} x). \quad (4.4)$$

Lemma 4.2. *There exists a constant $N = N(\alpha, d)$ such that*
(i) for any $t > a$ and $\eta > 0$,

$$\int_a^t \int_{|y| \geq \eta} |K(t-s, y)| dy ds \leq N(t-a) \eta^{-2/\alpha}; \quad (4.5)$$

(ii) for any $t > \tau > a$,

$$\int_{-\infty}^a \int_{\mathbf{R}^d} |K(t-s, y) - K(\tau-s, y)| dy ds \leq N \frac{t-\tau}{\tau-a}; \quad (4.6)$$

(iii) for any $t > a$ and $x \in \mathbf{R}^d$,

$$\int_{-\infty}^a \int_{\mathbf{R}^d} |K(t-s, x+y) - K(t-s, y)| dy ds \leq N|x|(t-a)^{-\alpha/2}. \quad (4.7)$$

Proof. First observe that

$$\int_{\mathbf{R}^{d+1}} \left(|y|^{2/\alpha} |K(1, y)| + |D_x K(1, y)| + |\partial_t K(1, y)| \right) dy < \infty,$$

which is an easy consequence of Lemma 3.2. By (4.2), it holds that

$$\begin{aligned} \int_a^t \int_{|y| \geq \eta} |K(t-s, y)| dy ds &= \int_a^t (t-s)^{-1-\frac{\alpha d}{2}} \int_{|y| \geq \eta} K(1, (t-s)^{-\alpha/2} y) dy ds \\ &= \int_a^t (t-s)^{-1} \int_{|y| \geq \eta(t-s)^{-\alpha/2}} K(1, y) dy ds \\ &\leq \left(\int_a^t \eta^{-2/\alpha} ds \right) \left(\int_{\mathbf{R}^d} |y|^{2/\alpha} K(1, y) dy \right) \\ &\leq N(t-a) \eta^{-2/\alpha}. \end{aligned}$$

Hence (i) is proved.

Next we prove (ii) and (iii) on the basis of the scaling property. By (4.3), (4.4), and the mean-value theorem, we have

$$\begin{aligned} &\int_{-\infty}^a \int_{\mathbf{R}^d} |K(t-s, y) - K(\tau-s, y)| dy ds \\ &\leq (t-\tau) \int_{-\infty}^a \int_{\mathbf{R}^d} \int_0^1 |\partial_s K(\theta t + (1-\theta)\tau - s, y)| d\theta dy ds \\ &\leq (t-\tau) \int_0^1 \left(\int_{-\infty}^a (\theta t + (1-\theta)\tau - s)^{-2} ds \right) \left(\int_{\mathbf{R}^d} |\partial_s K(1, y)| dy \right) d\theta \\ &\leq (t-\tau) \left(\int_{-\infty}^a (\tau-s)^{-2} ds \right) \leq N \frac{t-\tau}{\tau-a}. \end{aligned}$$

Also,

$$\begin{aligned}
& \int_{-\infty}^a \int_{\mathbf{R}^d} |K(t-s, y+x) - K(t-s, y)| dy ds \\
& \leq |x| \int_{-\infty}^a \int_{\mathbf{R}^d} \int_0^1 |D_y K(t-s, y+\theta x)| d\theta dy ds \\
& \leq |x| \int_{t-a}^{\infty} \int_{\mathbf{R}^d} |D_y K(s, y)| dy ds \\
& \leq |x| \int_{t-a}^{\infty} \int_{\mathbf{R}^d} |1_{t>0} t^{-1-\frac{\alpha}{2}} D_y K(1, y)| dy ds \leq N|x|(t-a)^{-\alpha/2}.
\end{aligned}$$

The lemma is proved. \square

Lemma 4.3. *Let $f \in L_2(\mathbf{R}^{d+1})$ and $f = 0$ on $Q_{2\delta}$. Then*

$$\int_{Q_\delta} \int_{Q_\delta} |\mathcal{G}f(t, x) - \mathcal{G}f(s, y)| ds dy dt dx \leq N(d, \alpha) \|f\|_{L_\infty(\mathbf{R}^{d+1})}. \quad (4.8)$$

Proof. First assume $f \in C_c^\infty(\mathbf{R}^{d+1})$. We claim that

$$\int_{Q_\delta} |\mathcal{G}f(t, x) - \mathcal{G}f(-\delta^{2/\alpha}, 0)| dt dx \leq N \|f\|_{L_\infty(\mathbf{R}^{d+1})}. \quad (4.9)$$

Let $(t, x) \in Q_\delta$. Then

$$\begin{aligned}
& |\mathcal{G}f(t, x) - \mathcal{G}f(-\delta^{2/\alpha}, 0)| \\
& \leq |\mathcal{G}f(t, x) - \mathcal{G}f(t, 0)| + |\mathcal{G}f(t, 0) - \mathcal{G}f(-\delta^{2/\alpha}, 0)| =: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

We consider \mathcal{I}_1 first.

$$\begin{aligned}
\mathcal{I}_1 &= \left| \int_{-\infty}^t \int_{\mathbf{R}^d} (K(t-s, x-y) - K(t-s, -y)) f(s, y) dy ds \right| \\
&= \left| \int_{-(2\delta)^{2/\alpha}}^t \int_{\mathbf{R}^d} \cdots dy ds + \int_{-\infty}^{-(2\delta)^{2/\alpha}} \int_{\mathbf{R}^d} \cdots dy ds \right| \\
&\leq \int_{-(2\delta)^{2/\alpha}}^t \int_{\mathbf{R}^d} |K(t-s, y)| |f(s, x-y)| dy ds \\
&\quad + \int_{-(2\delta)^{2/\alpha}}^t \int_{\mathbf{R}^d} |K(t-s, y)| |f(s, -y)| dy ds \\
&\quad + \int_{-\infty}^{-(2\delta)^{2/\alpha}} \int_{\mathbf{R}^d} |K(t-s, x-y) - K(t-s, -y)| |f(s, y)| dy ds \\
&=: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}.
\end{aligned}$$

Note that if $-(2\delta)^{2/\alpha} < s \leq t < \delta^{2/\alpha}$ and $|y| \leq \delta$, then

$$f(s, x-y) = 0, \quad f(s, -y) = 0, \quad (4.10)$$

because $|x - y| \leq 2\delta$, $|y| \leq \delta$, and $f = 0$ on $Q_{2\delta}$. Hence by (4.5), $\mathcal{I}_{11} + \mathcal{I}_{12}$ is less than or equal to

$$\begin{aligned} & N\|f\|_{L_\infty(\mathbf{R}^{d+1})} \int_{-(2\delta)^{2/\alpha}}^t \int_{|y| \geq \delta} |K(t-s, y)| dy ds \\ & \leq N\|f\|_{L_\infty(\mathbf{R}^{d+1})} (t + (2\delta)^{2/\alpha}) \delta^{-2/\alpha} \leq N\|f\|_{L_\infty(\mathbf{R}^{d+1})}. \end{aligned}$$

Also, by (4.7) we have

$$\begin{aligned} \mathcal{I}_{13} & \leq N\|f\|_{L_\infty(\mathbf{R}^{d+1})} \int_{-\infty}^{-(2\delta)^{2/\alpha}} \int_{\mathbf{R}^d} |K(t-s, x-y) - K(t-s, -y)| dy ds \\ & \leq N\|f\|_{L_\infty(\mathbf{R}^{d+1})} |x| (t + (2\delta)^{2/\alpha})^{-\alpha/2} \leq N\|f\|_{L_\infty(\mathbf{R}^{d+1})}. \end{aligned}$$

Next, we consider \mathcal{I}_2 . Note that

$$\begin{aligned} \mathcal{I}_2 & = \left| \mathcal{G}f(t, 0) - \mathcal{G}f(-\delta^{2/\alpha}, 0) \right| \\ & = \left| \int_{-\infty}^t \int_{\mathbf{R}^d} K(t-s, y) f(s, -y) dy ds - \int_{-\infty}^{-\delta^{2/\alpha}} \int_{\mathbf{R}^d} K(-\delta^{2/\alpha}-s, -y) f(s, -y) dy ds \right| \\ & \leq \left| \int_{-\infty}^t \int_{\mathbf{R}^d} K(t-s, y) f(s, -y) ds dy - \int_{-\infty}^{-\delta^{2/\alpha}} \int_{\mathbf{R}^d} K(t-s, y) f(s, -y) dy ds \right| \\ & + \left| \int_{-\infty}^{-\delta^{2/\alpha}} \int_{\mathbf{R}^d} [K(t-s, y) - K(-\delta^{2/\alpha}-s, y)] f(s, -y) dy ds \right| =: \mathcal{I}_{21} + \mathcal{I}_{22}. \end{aligned}$$

Recall (4.10). Thus by (4.5), we have

$$\begin{aligned} \mathcal{I}_{21} & \leq \int_{-\delta^{2/\alpha}}^t \int_{\mathbf{R}^d} |K(t-s, y) f(s, -y)| dy ds \\ & \leq \|f\|_{L_\infty(\mathbf{R}^{d+1})} \int_{-\delta^{2/\alpha}}^t \int_{|y| \geq \delta} |K(t-s, y)| dy ds \\ & \leq N\|f\|_{L_\infty(\mathbf{R}^{d+1})} (t + \delta^{2/\alpha}) \delta^{-2/\alpha} \leq N\|f\|_{L_\infty(\mathbf{R}^{d+1})}. \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{I}_{22} & \leq \int_{-(2\delta)^{2/\alpha}}^{-\delta^{2/\alpha}} \int_{\mathbf{R}^d} |K(t-s, y) - K(-\delta^{2/\alpha}-s, y)| |f(s, -y)| dy ds \\ & + \|f\|_{L_\infty(\mathbf{R}^{d+1})} \int_{-\infty}^{-(2\delta)^{2/\alpha}} \int_{\mathbf{R}^d} |K(t-s, y) - K(-\delta^{2/\alpha}-s, y)| dy ds \\ & =: \mathcal{I}_{221} + \mathcal{I}_{222}. \end{aligned}$$

By (4.5) again, we have

$$\begin{aligned} \mathcal{I}_{221} & \leq \|f\|_{L_\infty(\mathbf{R}^{d+1})} \int_{-(2\delta)^{2/\alpha}}^t \int_{|t| \geq \delta} |K(t-s, t)| dy ds \\ & + \|f\|_{L_\infty(\mathbf{R}^{d+1})} \int_{-(2\delta)^{2/\alpha}}^{-\delta^{2/\alpha}} \int_{|y| \geq \delta} |K(-\delta^{2/\alpha}-s, y)| dy ds \\ & \leq N\|f\|_{L_\infty(\mathbf{R}^{d+1})}. \end{aligned}$$

On the other hand, by (4.6) we obtain

$$\mathcal{I}_{222} \leq \frac{t + \delta^{2/\alpha}}{-\delta^{2/\alpha} + (2\delta)^{2/\alpha}} \|f\|_{L_\infty(\mathbf{R}^{d+1})} \leq N \|f\|_{L_\infty(\mathbf{R}^{d+1})}.$$

Hence (4.9) is proved and this obviously implies (4.8) for $f \in C_c^\infty(\mathbf{R}^{d+1})$.

Now we consider the general case, that is $f \in L_2(\mathbf{R}^{d+1})$. We choose a sequence of functions $f_n \in C_c^\infty(\mathbf{R}^{d+1})$ such that $f_n = 0$ on $Q_{2\delta}$, $\mathcal{G}f_n \rightarrow \mathcal{G}f$ (a.e.), and $\|f_n\|_{L_\infty(\mathbf{R}^{d+1})} \leq \|f\|_{L_\infty(\mathbf{R}^{d+1})}$. Then by Fatou's lemma,

$$\begin{aligned} & \int_{Q_\delta} \int_{Q_\delta} |\mathcal{G}f(t, x) - \mathcal{G}f(s, y)| \, dt ds dx dy \\ & \leq \liminf_{n \rightarrow \infty} \int_{Q_\delta} \int_{Q_\delta} |\mathcal{G}f_n(t, x) - \mathcal{G}f_n(s, y)| \, dt ds dx dy \\ & \leq N \liminf_{n \rightarrow \infty} \|f_n\|_{L_\infty(\mathbf{R}^{d+1})} \leq N \|f\|_{L_\infty(\mathbf{R}^{d+1})}. \end{aligned}$$

The lemma is proved. \square

Theorem 4.4. (i) For any $f \in L_2(\mathbf{R}^{d+1}) \cap L_\infty(\mathbf{R}^{d+1})$,

$$\|\mathcal{G}f\|_{BMO(\mathbf{R}^{d+1})} \leq N(d, \alpha) \|f\|_{L_\infty(\mathbf{R}^{d+1})}. \quad (4.11)$$

(ii) For any $p, q \in (1, \infty)$ and $f \in C_c^\infty(\mathbf{R}^{d+1})$,

$$\|\mathcal{G}f\|_{L_q(\mathbf{R}, L_p)} \leq N(d, p, q, \alpha) \|f\|_{L_q(\mathbf{R}, L_p)}. \quad (4.12)$$

Proof. (i) It suffices to prove that for each $Q = Q_\delta(t_0, x_0)$ the following holds :

$$\int_Q |\mathcal{G}f(t, x) - (\mathcal{G}f)_Q| \, dt dx \leq N \|f\|_{L_\infty(\mathbf{R}^{d+1})}. \quad (4.13)$$

Due to the translation invariant property of the operator \mathcal{G} , we may assume that $(t_0, x_0) = 0$. Thus

$$Q = Q_\delta = (-\delta^{2/\alpha}, \delta^{2/\alpha}) \times B_\delta.$$

Take $\zeta \in C_c^\infty(\mathbf{R}^{d+1})$ such that $\zeta = 1$ on $Q_{2\delta}$ and $\zeta = 0$ outside $Q_{3\delta}$. Then

$$\begin{aligned} & \int_Q |\mathcal{G}f(t, x) - (\mathcal{G}f)_Q| \, dt dx \\ & \leq \int_Q |\mathcal{G}(\zeta f) - (\mathcal{G}(\zeta f))_Q| \, dt dx + \int_Q |\mathcal{G}((1 - \zeta)f) - (\mathcal{G}((1 - \zeta)f))_Q| \, dt dx \\ & \leq 2 \int_Q |\mathcal{G}(\zeta f)| \, dt dx + \int_Q \int_Q |\mathcal{G}((1 - \zeta)f)(t, x) - \mathcal{G}((1 - \zeta)f)(s, y)| \, ds dy dt dx. \end{aligned}$$

Thus (4.13) comes from Lemma 4.1 and Lemma 4.3.

(ii) **Step 1.** We prove (4.12) for the case $q = p$. First we assume $q = p \geq 2$. For a measurable function $h(t, x)$ on \mathbf{R}^{d+1} , we define the maximal function

$$\mathcal{M}h(t, x) = \sup_{Q \in \mathbb{Q}} \int_Q |f(r, z)| \, dr dz,$$

and the sharp function

$$h^\sharp(t, x) = \sup_{Q \in \mathbb{Q}} \int_Q |f(r, z) - f_Q| \, dr dz.$$

Then by the Fefferman-Stein theorem and the Hardy-Littlewood maximal theorem

$$\|h^\sharp\|_{L_p(\mathbf{R}^{d+1})} \sim \|\mathcal{M}h\|_{L_p(\mathbf{R}^{d+1})} \sim \|h\|_{L_p(\mathbf{R}^{d+1})}. \quad (4.14)$$

Combining Lemma 3.6 with (4.14), we get for any $f \in L_2(\mathbf{R}^{d+1}) \cap L_\infty(\mathbf{R}^{d+1})$,

$$\|(\mathcal{G}f)^\sharp\|_{L_2(\mathbf{R}^{d+1})} \leq N\|f\|_{L_2(\mathbf{R}^{d+1})}.$$

Also, by (4.11)

$$\|(\mathcal{G}f)^\sharp\|_{L_\infty(\mathbf{R}^{d+1})} \leq N\|f\|_{L_\infty(\mathbf{R}^{d+1})}.$$

Note that the map $f \rightarrow (\mathcal{G}f)^\sharp$ is subadditive since \mathcal{G} is a linear operator. Hence by a version of the Marcinkiewicz interpolation theorem (see e.g. [17, Lemma 3.4]), for any $p \in [2, \infty)$ there exists a constant N such that

$$\|(\mathcal{G}f)^\sharp\|_{L_p(\mathbf{R}^{d+1})} \leq N\|f\|_{L_p(\mathbf{R}^{d+1})},$$

for all $f \in L_2(\mathbf{R}^{d+1}) \cap L_\infty(\mathbf{R}^{d+1})$. Finally by the Fefferman-Stein theorem, we get

$$\|\mathcal{G}f\|_{L_p(\mathbf{R}^{d+1})} \leq N(d, \alpha, p)\|f\|_{L_p(\mathbf{R}^{d+1})}.$$

Therefore (4.12) is proved for $q = p \in [2, \infty)$.

For $p \in (1, 2)$, we use the duality argument. Let $f, g \in C_c^\infty(\mathbf{R}^{d+1})$ and p' be the conjugate of p , i.e. $p' = \frac{p}{p-1} \in (2, \infty)$. By the integration by parts, the change of variable, and Fubini's theorem,

$$\begin{aligned} & \int_{\mathbf{R}^{d+1}} g(t, x) \mathcal{G}f(t, x) dx dt \\ &= \int_{\mathbf{R}^{d+1}} \Delta g(t, x) \left(\int_{\mathbf{R}^{d+1}} 1_{t>s} q(t-s, x-y) f(s, y) ds dy \right) dx dt \\ &= \int_{\mathbf{R}^{d+1}} f(-s, -y) \left(\int_{\mathbf{R}^{d+1}} 1_{s>t} q(s-t, y-x) \Delta g(-t, -x) dx dt \right) ds dy \\ &= \int_{\mathbf{R}^{d+1}} f(-s, -y) \mathcal{G}\tilde{g}(s, y) ds dy \end{aligned} \quad (4.15)$$

where $\tilde{g}(t, x) = g(-t, -x)$. Then by Hölder's inequality (also recall $2 < p' < \infty$),

$$\left| \int_{\mathbf{R}^{d+1}} g(t, x) \mathcal{G}f(t, x) dx dt \right| \leq N\|f\|_{L_p} \|\mathcal{G}\tilde{g}\|_{L_{p'}} \leq N\|f\|_{L_p} \|g\|_{L_{p'}}.$$

Since $g \in C_c^\infty(\mathbf{R}^{d+1})$ is arbitrary, (4.11) is proved for $q = p \in (1, 2)$ as well.

Step 2. Now we prove (4.12) for general $p, q \in (1, \infty)$. For each $(t, s) \in \mathbf{R}^2$, we define the operator $\mathcal{K}(t, s)$ as follows:

$$\mathcal{K}(t, s)f(x) := \int_{\mathbf{R}^d} K(t-s, x-y)f(y)dy, \quad f \in C_c^\infty.$$

Let $p \in (1, \infty)$ and $(t, s) \in \mathbf{R}^2$. Then by the mean-value theorem, Lemma 3.2, and (4.2),

$$\begin{aligned} \|\mathcal{K}(t, s)f\|_{L_p} &= \left\| \int_{\mathbf{R}^d} K(t-s, x-y)f(y)dy \right\|_{L_p} \\ &\leq \|f\|_{L_p} \int_{\mathbf{R}^d} |K(t-s, y)|dy \leq N(d, \alpha)(t-s)^{-1}\|f\|_{L_p}. \end{aligned}$$

Hence the operator $\mathcal{K}(t, s)$ is uniquely extendible to L_p for $t \neq s$. Denote

$$Q := [t_0, t_0 + \delta), \quad Q^* := [t_0 - \delta, t_0 + 2\delta).$$

Note that for $t \notin Q^*$ and $s, r \in Q$, we have

$$|s - r| \leq \delta, \quad |t - (t_0 + \delta)| \geq \delta,$$

and recall $K(t - s, x - y) = 0$ if $t \leq s$. Thus by Lemma 3.2 and (4.3), it holds that

$$\begin{aligned} & \|\mathcal{K}(t, s) - \mathcal{K}(t, r)\|_{\mathbf{L}(L_p)} \\ &= \sup_{\|f\|_{L_p}=1} \left\| \int_{\mathbf{R}^d} \{K(t - s, x - y) - K(t - r, x - y)\} f(y) dy \right\|_{L_p} \\ &\leq \sup_{\|f\|_{L_p}=1} \|f\|_{L_p} \int_{\mathbf{R}^d} |K(t - s, x) - K(t - r, x)| dx \\ &= \int_{\mathbf{R}^d} |K(t - s, x) - K(t - r, x)| dx \leq \frac{N(d, \alpha) |s - r|}{(t - (t_0 + \delta))^2}, \end{aligned}$$

where $\mathbf{L}(L_p)$ denotes the bounded linear operator norm on L_p . Therefore,

$$\begin{aligned} & \int_{\mathbf{R} \setminus Q^*} \|\mathcal{K}(t, s) - \mathcal{K}(t, r)\|_{L(L_p)} dt \leq \int_{\mathbf{R} \setminus Q^*} \frac{|s - r|}{(t - (t_0 + \delta))^2} dt \\ &\leq N |s - r| \int_{|t - (t_0 + \delta)| \geq \delta} \frac{1}{(t - (t_0 + \delta))^2} dt \leq N \delta \int_{\delta}^{\infty} t^{-2} dt \leq N. \end{aligned}$$

Furthermore, by following the proof of Theorem 1.1 of [20], one can easily check that for almost every t outside of the support of $f \in C_c^\infty(\mathbf{R}, L_p)$,

$$\mathcal{G}f(t, x) = \int_{-\infty}^{\infty} \mathcal{K}(t, s) f(s, x) ds$$

where \mathcal{G} denotes the unique extension on $L_p(\mathbf{R}^{d+1})$ which is verified in Step 1. Hence by the Banach space-valued version of the Calderón-Zygmund theorem [20, Theorem 4.1], our assertion is proved for $1 < q \leq p$.

For the remaining case that $1 < p < q < \infty$, we use the duality argument. Define $p' = p/(p - 1)$ and $q' = q/(q - 1)$. Since $1 < q' < p'$, by (4.15) and Hölder's inequality,

$$\begin{aligned} \int_{\mathbf{R}^{d+1}} g(t, x) \mathcal{G}f(t, x) dx dt &= \int_{\mathbf{R}} \left(\int_{\mathbf{R}^d} f(-s, -y) \mathcal{G}\tilde{g}(s, y) dy \right) ds \\ &\leq \int_{\mathbf{R}} \|f(-s, \cdot)\|_{L_p} \|\mathcal{G}\tilde{g}(s, \cdot)\|_{L_{p'}} ds \\ &\leq N(d, \alpha, p) \|f\|_{L_q(\mathbf{R}, L_p)} \|g\|_{L_{q'}(\mathbf{R}, L_{p'})} \end{aligned}$$

for any $f, g \in C_c^\infty(\mathbf{R}^{d+1})$. Since g is arbitrary, we have

$$\|\mathcal{G}f\|_{L_q(\mathbf{R}, L_p)} \leq N(d, \alpha, p) \|f\|_{L_q(\mathbf{R}, L_p)}.$$

The theorem is proved. \square

5. PROOF OF THEOREM 2.9

First we consider the solvability of the model equation.

Lemma 5.1. *Theorem 2.9 holds for the equation $\partial_t^\alpha u = \Delta u + f_0$ with $N_0 = N_0(p, q, \alpha, T)$.*

Proof. First we prove the uniqueness. Suppose that $u \in \mathbb{H}_{q,p,0}^{\alpha,2}(T)$ and $\partial_t^\alpha u = \Delta u$. Due to Lemma 2.6 (iii), there exists a $u_n \in C_c^\infty(\mathbf{R}^{d+1})$ such that $u_n \rightarrow u$ in $\mathbb{H}_{q,p,0}^{\alpha,2}(T)$. Due to Lemma 3.5 (i),

$$u_n(t, x) = \int_0^t \int_{\mathbf{R}^d} q(t-s, x-y) f_n(s, y) dy ds,$$

where $f_n := \partial_t^\alpha u_n - \Delta u_n$. Since $f_n \rightarrow 0$ in $\mathbb{L}_p^q(T)$, we obtain the uniqueness from Lemma 2.6 (iv) and Theorem 4.4.

For the existence and (2.12), first assume $f \in C_c^\infty(\mathbf{R}_+^{d+1})$ and define $u = \mathcal{G}f$. Then all the claims follows from Lemma 3.5(ii), Theorem 4.4, and (2.8). For general f one can consider an approximation $f_n \rightarrow f$, and above arguments show that $\mathcal{G}f_n$ is a Cauchy sequence in $\mathbb{H}_{q,p,0}^{\alpha,2}(T)$ and the limit becomes a solution of $\partial_t^\alpha u = \Delta u + f$. \square

The following two results will be used later when we extend results proved for small T to the case when T is arbitrary.

Lemma 5.2. *Let $u \in \mathbb{H}_{q,p,0}^{\alpha,2}(\tilde{T})$ and $\tilde{T} \leq T$. Then there exists $\tilde{u} \in \mathbb{H}_{q,p,0}^{\alpha,2}(T)$ such that $\tilde{u}(t) = u(t)$ for all $t \leq \tilde{T}$, and*

$$\|\tilde{u}\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)} \leq N_0 \|u\|_{\mathbb{H}_{q,p}^{\alpha,2}(\tilde{T})}, \quad (5.1)$$

where N_0 is from Lemma 5.1 and is independent of \tilde{T} .

Proof. Denote $f = \partial_t^\alpha u$, and let $\tilde{u} \in \mathbb{H}_{q,p,0}^{\alpha,2}(T)$ be the solution of

$$\tilde{u}_t^\alpha = \Delta \tilde{u} + (f - \Delta u) 1_{t \leq \tilde{T}}, \quad t \leq T.$$

Then by Lemma 5.1,

$$\|\tilde{u}\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)} \leq N_0 \|(f - \Delta u) 1_{t \leq \tilde{T}}\|_{\mathbb{L}_{q,p}(T)} \leq N_0 \|u\|_{\mathbb{H}_{q,p}^{\alpha,2}(\tilde{T})}.$$

Next observe that for $t \leq \tilde{T}$,

$$\partial_t^\alpha (\tilde{u} - u) = \Delta \tilde{u} + f - \Delta u - f = \Delta (\tilde{u} - u).$$

It follows from Lemma 5.1 that $\tilde{u} = u$ for $t \leq \tilde{T}$. The lemma is proved. \square

Lemma 5.3. *Let $0 < \tilde{T} < T$ and $u, \tilde{u} \in \mathbb{H}_{q,p,0}^{\alpha,2}(T)$. Assume that*

$$u(t) = \tilde{u}(t) \quad t \leq \tilde{T} \quad (a.e.). \quad (5.2)$$

Then $\bar{u}(t) := u(\tilde{T} + t) - \tilde{u}(\tilde{T} + t) \in \mathbb{H}_{q,p,0}^{\alpha,2}(T - \tilde{T})$.

Proof. We take $u^{\varepsilon_1, \varepsilon_2, \varepsilon_3}$ from the proof of Lemma 2.6 (iii) which is defined as

$$u^{\varepsilon_1, \varepsilon_2, \varepsilon_3}(t, x) = \eta(t) \eta_3(\varepsilon_3 x) \int_{\mathbf{R}^d} \int_0^\infty u(s, y) \eta_{1, \varepsilon_1}(t-s) \eta_{2, \varepsilon_2}(x-y) ds dy.$$

As shown before, for any $\varepsilon > 0$ it holds that

$$\|u - u^{\varepsilon_1, \varepsilon_2, \varepsilon_3}\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)} \leq \varepsilon$$

if $\varepsilon_1, \varepsilon_2$, and ε_3 are small enough. Hence we can take a sequence (a_n, b_n, c_n) so that

$$\|u - u^{a_n, b_n, c_n}\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and

$$\|\tilde{u} - \tilde{u}^{a_n, b_n, c_n}\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Observe that

$$u^{a_n, b_n, c_n}(t, x) = \tilde{u}^{a_n, b_n, c_n}(t, x) \quad \forall t \leq (\tilde{T} + a_n) \wedge T$$

due to (5.2) and the fact that $\eta_{1, \varepsilon_1} \in C_c^\infty((\varepsilon_1, 2\varepsilon_1))$. Thus

$$\bar{u}_n(t, x) := u^{a_n, b_n, c_n}(\tilde{T} + t, x) - \tilde{u}^{a_n, b_n, c_n}(\tilde{T} + t, x)$$

is a defining sequence of \bar{u} such that

$$\bar{u}_n(0, x) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} \bar{u}_n(0, x) = 0 \quad \forall x \in \mathbf{R}^d, \quad \forall n \in \mathbb{N}.$$

Therefore the lemma is proved. \square

Proof of Theorem 2.9.

Step 1. Denote $\bar{f} = b^i u_{x^i} + cu + f(u)$. Then

$$\begin{aligned} \|\bar{f}(u) - \bar{f}(v)\|_{L_p} &\leq N\|u - v\|_{H_p^1} + \|f(u) - f(v)\|_{L_p} \\ &\leq \varepsilon\|u - v\|_{H_p^2} + K\|u - v\|_{L_p} + \|f(u) - f(v)\|_{L_p}. \end{aligned}$$

Thus considering \bar{f} in place of f we may assume $b^i = c = 0$.

Step 2. Let $f = f_0$ be independent of u . Assume that a^{ij} are independent of (t, x) . In this case obviously we may assume $a^{ij} = \delta^{ij}$, and therefore the results follow from Lemma 5.1.

Step 3. Let $f = f_0$. Suppose that Theorem 2.9 holds with some matrix $\bar{a} = (\bar{a}^{ij}(t, x))$ in place of $(a^{ij}(t, x))$. We prove that there exists $\varepsilon_0 = \varepsilon_0(N_0) > 0$ so that if

$$\sup_{(t, x)} |a^{ij}(t, x) - \bar{a}^{ij}(t, x)| \leq \varepsilon_0,$$

then Theorem 2.9 also holds for $(a^{ij}(t, x))$ with $2N_0$ in place of N_0 . To prove this, due to the method of continuity, we only need to prove that (2.12) holds given that a solution u of (2.11) already exists. Note that u satisfies

$$\partial_t^\alpha u = \bar{a}^{ij} u_{x^i x^j} + \bar{f}, \quad \bar{f} = f + (a^{ij} - \bar{a}^{ij}) u_{x^i x^j}.$$

Hence by the assumption,

$$\begin{aligned} \|u\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)} &\leq N_0 \|(a^{ij} - \bar{a}^{ij}) u_{x^i x^j}\|_{\mathbb{L}_{q,p}(T)} + N_0 \|f\|_{\mathbb{L}_{q,p}(T)} \\ &\leq N_0 \sup_{t, x} |a^{ij} - \bar{a}^{ij}| \|u\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)} + N_0 \|f\|_{\mathbb{L}_{q,p}(T)}. \end{aligned}$$

Thus it is enough to take $\varepsilon_0 = (2N_0)^{-1}$.

Step 4. Let $a^{ij} = a^{ij}(x)$ depend only on x and $f = f_0$. In this case one can repeat the classical perturbation arguments to prove the claims. Below we give a detail for the sake of the completeness. As before we only need to prove that there exists a constant N_0 independent of u such that (2.12) holds given that u is a solution. By Steps 2 and 3, there exists $\varepsilon_0 > 0$ depending only on p, q, α, K , and T such that the theorem holds true if there exists any point $x_0 \in \mathbf{R}^d$ such that

$$\sup_x |a^{ij}(x) - a^{ij}(x_0)| \leq \varepsilon_0. \quad (5.3)$$

Recall that a^{ij} is uniformly continuous. Let $\delta_0 < 1$ be a constant depending on ε_0 such that

$$|a^{ij}(x) - a^{ij}(y)| \leq \varepsilon_0/2, \quad \text{if } |x - y| < 4\delta_0.$$

Choose a partition of unity $\phi_n, n = 1, 2, \dots$ so that $\phi_n = \phi(\frac{x-x_n}{\delta_0})$ for some $x_n \in \mathbf{R}^d$ and $\phi \in C_c^\infty(B_2(0))$ satisfying $0 \leq \phi \leq 1$ and $\phi = 1$ for $|x| \leq 1$. Denote $\bar{\phi}_n = \phi(\frac{x-x_n}{2\delta_0})$. Then $\bar{\phi}_n = 1$ on the support of ϕ_n , and $u_n = \phi_n u$ satisfies

$$\partial_t^\alpha u_n = a_n^{ij}(u_n)_{x^i x^j} + f_n$$

where

$$\begin{aligned} a_n^{ij} &= \bar{\phi}_n a^{ij}(x) + (1 - \bar{\phi}_n) a^{ij}(x_n), \\ f_n &= f \phi_n - 2a^{ij} u_{x^i}(\phi_n)_{x^j} - a^{ij} u(\phi_n)_{x^i x^j}. \end{aligned}$$

It is easy to check (a_n^{ij}) satisfies (2.9) and (5.3) with x_n in place of x_0 . By [19, Lemma 6.7] and Step 3, if $p = q$ then for any $t \leq T$,

$$\|u\|_{\mathbb{H}_{q,p}^{0,2}(t)}^q \sim \sum_{n=1}^{\infty} \|u \phi_n\|_{\mathbb{H}_{q,p}^{0,2}(t)}^q \leq N_0 \sum_{n=1}^{\infty} \|f_n\|_{\mathbb{L}_{q,p}(t)}^q \quad (5.4)$$

$$\begin{aligned} &\leq N \sum_{n=1}^{\infty} \left[\|u_x(\phi_n)_x\|_{\mathbb{L}_{q,p}(t)}^q + \|u(\phi_n)_{xx}\|_{\mathbb{L}_{q,p}(t)}^q + \|f \phi_n\|_{\mathbb{L}_{q,p}(t)}^q \right] \\ &\leq N \|u\|_{\mathbb{H}_{q,p}^{0,1}(t)}^q + N \|f\|_{\mathbb{L}_{q,p}(t)}^q. \end{aligned} \quad (5.5)$$

The last inequality above is also from Lemma 6.7 of [19]. We emphasize that equivalence relation in (5.4) and inequality (5.5) hold in general only if $p = q$ or only finite ϕ_n are non-zero functions. Hence, if $q \neq p$ then we take sufficiently large $R, M > 0$ so that $\sum_{n=1}^M \phi_n(x) = 1$ on B_R and vanishes for $|x| \geq 2R$, and the oscillation of a^{ij} on the complement of $B_{R/2}$ is less than $\varepsilon_0/2$. Denote $\phi_0 = 1 - \sum_{n=1}^M \phi_n$. Then one can repeat the above calculations and use the relation $\|u\|_{H_p^1} \leq \varepsilon \|u\|_{H_p^2} + N \|u\|_{L_p}$ to conclude that for all $t \leq T$

$$\begin{aligned} \|u\|_{\mathbb{H}_{q,p}^{0,2}(t)}^q &\leq N \|u\|_{\mathbb{L}_{q,p}(t)}^q + N \|f\|_{\mathbb{L}_{q,p}(t)}^q \\ &\leq N \int_0^t \left[\int_0^s (s-r)^{-1+\alpha} (\|\partial_t^\alpha u(r)\|_{H_p^2} + \|f(r)\|_{L_p}) dr \right]^q ds + N \|f\|_{\mathbb{L}_{q,p}(t)}^q \\ &\leq N \int_0^t (t-s)^{-1+\alpha} \|u\|_{\mathbb{H}_{q,p}^{0,2}(s)}^q ds + N \|f\|_{\mathbb{L}_{q,p}(T)}^q, \end{aligned}$$

where Theorem 2.6 (iv) is used in the second inequality. Hence by a version of Gronwall's lemma (see e.g. [45, Corollary 2]) we get

$$\|u\|_{\mathbb{H}_{q,p}^{0,2}(T)}^q \leq N \|f\|_{\mathbb{L}_{q,p}(T)}^q. \quad (5.6)$$

From equation (2.11), we easily conclude that

$$\|\partial_t^\alpha u\|_{\mathbb{L}_{q,p}(T)} \leq N \|u\|_{\mathbb{H}_{q,p}^{0,2}(T)} + \|f\|_{\mathbb{L}_{q,p}(T)},$$

and therefore (5.6) certainly leads to the a priori estimate.

Step 5. Let $f_0 = f$, and a^{ij} be uniformly continuous in (t, x) . As before, we only need to prove the a priori estimate (2.12).

Let N_0 be the constant from Step 4 so that a priori estimate (2.12) holds whenever a^{ij} are independent of t . Take ε_0 from Step 3 corresponding to this N_0 . We will apply Step 3 with $\bar{a}^{ij} = a^{ij}(t_0, x)$ for some t_0 .

Take $\kappa > 0$ so that

$$|a^{ij}(t, x) - a^{ij}(s, x)| \leq \varepsilon_0/2, \quad \text{if } |t - s| \leq 2\kappa.$$

Also take an integer N so that $T/N \leq \kappa$, and denote $\tilde{T}_i = iT/N$. By Steps 3 and 4 applied with $\bar{a}^{ij}(t, x) = a^{ij}(0, x)$, (2.12) holds with T_1 and $2N_0$ in place of T and N_0 respectively. Now we use the induction. Suppose that the a priori estimate (2.12) holds for $\tilde{T}_k < T$ with N_0 independent of u . This constant N_0 may depend on k . Take \tilde{u} from Lemma 5.2 corresponding to $\tilde{T} = \tilde{T}_k$. Denote

$$\bar{u}(t, x) = (u - \tilde{u})(\tilde{T} + t, x), \quad \bar{f}(t, x) = f(\tilde{T} + t, x).$$

Then one can easily check that \bar{u} satisfies

$$\partial_t^\alpha \bar{u} = a^{ij}(\tilde{T} + t, x) \bar{u}_{x^i x^j} + \bar{f} + (a^{ij}(\tilde{T} + t, x) - \delta^{ij}) \tilde{u}_{x^i x^j}(\tilde{T} + t, x), \quad t \leq T - \tilde{T}.$$

Due to Lemma 5.3, \bar{u} is contained in $\mathbb{H}_{q,p,0}^{\alpha,n}(T)$. Thus by the result of Steps 3 and 4 with $\bar{a}^{ij} = a^{ij}(\tilde{T}_k, x)$, we have

$$\begin{aligned} \|\bar{u}\|_{\mathbb{H}_{q,p}^{\alpha,2}(T_1)}^q &\leq (2N_0)^q \int_{\tilde{T}_k}^{\tilde{T}_{k+1}} \|f + (a^{ij} - \delta^{ij}) \tilde{u}_{x^i x^j}\|_{L_p}^q dt \\ &\leq N \|\tilde{u}\|_{\mathbb{H}_{q,p}^{0,2}(T)}^q + N \|f\|_{\mathbb{L}_{q,p}(T)}^q \\ &\leq N \|u\|_{\mathbb{H}_{q,p}^{\alpha,2}(\tilde{T}_k)}^q + \|f\|_{\mathbb{L}_{q,p}(T)}^q \leq N \|f\|_{\mathbb{L}_{q,p}(T)}^q, \end{aligned}$$

where the third inequality is due to (5.1) and the last inequality is from the assumption. Hence

$$\|u\|_{\mathbb{H}_{q,p}^{\alpha,2}(\tilde{T}_{k+1})}^q \leq \|\tilde{u}\|_{\mathbb{H}_{q,p}^{\alpha,2}(\tilde{T}_{k+1})}^q + \|\bar{u}\|_{\mathbb{H}_{q,p}^{\alpha,2}(\tilde{T}_1)}^q \leq N \|f\|_{\mathbb{L}_{q,p}(T)}^q.$$

As the induction goes through, the a priori estimate (2.12) is proved. We emphasize that for each k the constant N_0 varies, however for each \tilde{T}_k we use the result in Step 4 and therefore the choice of \tilde{T}_{k+1} (or the difference $|\tilde{T}_{k+1} - \tilde{T}_k|$) does not depend on k , and therefore we can reach up to T by finite such steps.

Step 6. Let $f_0 = f$ and $a^{ij} = \sum_{k=1}^\ell a_k^{ij}(t, x) I_{(T_{k-1}, T_k]}(t)$. If $\ell = 1$, (2.12) comes directly from Step 5. If $\ell > 1$, we use the induction argument used in Step 5. The only difference is that to estimate the solution on $[T_k, T_{k+1})$ we use the result of Step 5, in place of the result in Step 4.

Step 7. The general non-linear case. We modify the proof of Theorem 5.1 of [19]. For each $u \in \mathbb{H}_{q,p,0}^{\alpha,2}(T)$ consider the equation

$$\partial_t^\alpha v = a^{ij} + f(u), \quad t \leq T.$$

By the above results, this equation has a unique solution $v \in \mathbb{H}_{q,p,0}^{\alpha,2}(T)$. By denoting $v = \mathcal{R}u$ we can define an operator $\mathcal{R} : \mathbb{H}_{q,p,0}^{\alpha,2}(T) \rightarrow \mathbb{H}_{q,p,0}^{\alpha,2}(T)$. By the results for the linear case, for each $t \leq T$,

$$\begin{aligned} \|\mathcal{R}u - \mathcal{R}v\|_{\mathbb{H}_{q,p}^{\alpha,2}(t)}^q &\leq N \|f(u) - f(v)\|_{\mathbb{L}_{q,p}(t)}^q \\ &\leq N \varepsilon^q \|u - v\|_{\mathbb{H}_{q,p}^{0,2}(t)}^q + N K_\varepsilon^q \|u - v\|_{\mathbb{L}_{q,p}(t)}^q \\ &\leq N_0 \varepsilon^q \|u - v\|_{\mathbb{H}_{q,p}^{\alpha,2}(t)}^q + N_1 \int_0^t (t-s)^{-1+\alpha} \|u - v\|_{\mathbb{H}_{q,p}^{\alpha,2}(s)}^q ds, \end{aligned}$$

where N_1 depends also on ε , and Theorem 2.6 (iv) is used in the last inequality. Next, we fix ε so that $\theta := N_0 \varepsilon^q < 1/4$. Then repeating the above inequality and using the identity

$$\begin{aligned} & \int_0^t (t-s_1)^{-1+\alpha} \int_0^{s_1} (s_1-s_2)^{-1+\alpha} \cdots \int_0^{s_{n-1}} (s_{n-1}-s_n)^{-1+\alpha} ds_n \cdots ds_1 \\ &= \frac{\{\Gamma(\alpha)\}^n}{\Gamma(n\alpha+1)} t^{n\alpha}, \end{aligned}$$

we get

$$\begin{aligned} & \|\mathcal{R}^m u - \mathcal{R}^m v\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)}^q \\ & \leq \sum_{k=0}^m \binom{m}{k} \theta^{m-k} (T^\alpha N_1)^k \frac{\{\Gamma(\alpha)\}^k}{\Gamma(k\alpha+1)} \|u-v\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)}^q \\ & \leq 2^m \theta^m \left[\max_k \left(\frac{(\theta^{-1} T^\alpha N_1 \Gamma(\alpha))^k}{\Gamma(k\alpha+1)} \right) \right] \|u-v\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)}^q \\ & \leq \frac{1}{2^m} N_2 \|u-v\|_{\mathbb{H}_{q,p}^{\alpha,2}(T)}^q. \end{aligned}$$

For the second inequality above we use $\sum_{k=0}^m \binom{m}{k} = 2^m$. It follows that if m is sufficiently large then \mathcal{R}^m is a contraction in $\mathbb{H}_{q,p,0}^{\alpha,2}(T)$, and this yields all the claims. The theorem is proved. \square

6. KERNELS p AND q

6.1. The kernel $p(t, x)$. In this subsection, we prove Lemma 3.1(i). In other words, we introduce a kernel $p(t, x)$ which is integrable with respect to x and satisfies

$$\mathcal{F}\{p(t, \cdot)\}(\xi) = E_\alpha(-t^\alpha |\xi|^2). \quad (6.1)$$

Let $\Gamma(z)$ denote the gamma function which can be defined (see [2, Section 1.1]) for $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ as

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{z(z+1) \cdots (z+n)}.$$

Note that $\Gamma(z)$ is a meromorphic function with simple poles at the nonpositive integers. From the definition, for $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$,

$$z\Gamma(z) = \Gamma(z+1). \quad (6.2)$$

By (6.2), one can easily check that for $k = 0, 1, 2, \dots$,

$$\text{Res}_{z=-k} \Gamma(z) = \lim_{z \rightarrow -k} (z+k) \Gamma(z) = \frac{(-1)^k}{k!}, \quad (6.3)$$

where $\text{Res}_{z=-k} \Gamma(z)$ denotes the residue of $\Gamma(z)$ at $z = -k$. It is also well-known (see. e.g. [2, Theorem 1.1.4]) that if $\Re[z] > 0$ and $\Re[\omega] > 0$ then

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \int_0^1 t^{z-1} (1-t)^{\omega-1} dt = \frac{\Gamma(z) \Gamma(\omega)}{\Gamma(z+\omega)}. \quad (6.4)$$

Using Stirling's approximation (see [2, Corollary 1.4.3] or [16, (1.2.3)])

$$\Gamma(z) \sim \sqrt{2\pi} e^{(z-\frac{1}{2}) \log z} e^{-z}, \quad |z| \rightarrow \infty, \quad (6.5)$$

one can easily show that for fixed $a \in \mathbf{R}$, $([16, (1.2.2)])$

$$|\Gamma(a + ib)| \sim \sqrt{2\pi} |b|^{a-\frac{1}{2}} e^{-a-\frac{\pi}{2}|b|}, \quad |b| \rightarrow \infty. \quad (6.6)$$

Let m, n, μ, ν be fixed integers satisfying $0 \leq m \leq \mu, 0 \leq n \leq \nu$. Assume that the complex parameters $c_1, \dots, c_\nu, d_1, \dots, d_\mu$ and positive real parameters $\gamma_1, \dots, \gamma_\nu, \delta_1, \dots, \delta_\mu$ are given so that $P_1 \cap P_2 = \emptyset$ where

$$P_1 := \left\{ -\frac{d_j + k}{\delta_j} \in \mathbb{C} : j \in \{1, \dots, m\}, k = 0, 1, 2, \dots \right\}$$

$$P_2 := \left\{ \frac{1 - c_j + k}{\gamma_j} \in \mathbb{C} : j \in \{1, \dots, n\}, k = 0, 1, 2, \dots \right\}.$$

If either $m = 0$ or $n = 0$, then by the definition $P_1 \cap P_2 = \emptyset$. For the above parameters, the Fox H-function $H(r)$ ($r > 0$) is defined as

$$H(r) := H_{\nu\mu}^{mn} \left[r \mid \begin{matrix} (c_1, \gamma_1) & \cdots & (c_n, \gamma_n) & (c_{n+1}, \gamma_{n+1}) & \cdots & (c_\nu, \gamma_\nu) \\ (d_1, \delta_1) & \cdots & (d_m, \delta_m) & (d_{m+1}, \delta_{m+1}) & \cdots & (d_\mu, \delta_\mu) \end{matrix} \right]$$

$$:= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(d_j + \delta_j z) \prod_{j=1}^n \Gamma(1 - c_j - \gamma_j z)}{\prod_{j=m+1}^\mu \Gamma(1 - d_j - \delta_j z) \prod_{j=n+1}^\nu \Gamma(c_j + \gamma_j z)} r^{-z} dz, \quad (6.7)$$

where the contour L is chosen appropriately depending on the parameters. Some special cases needed in our setting are specified below. In this article, we additionally assume that parameters c_1, \dots, c_ν and d_1, \dots, d_μ are real and

$$\sum_{j=1}^\mu \delta_j - \sum_{i=1}^\nu \gamma_i > 0, \quad \sum_{i=1}^n \gamma_i - \sum_{i=n+1}^\nu \gamma_i + \sum_{j=1}^m \delta_j - \sum_{j=m+1}^\mu \delta_j > 0. \quad (6.8)$$

Under (6.8), we can choose the contour L of two different types. Hankel contour L_h is a loop starting at the point $-\infty + i\rho_1$ and ending at the point $-\infty + i\rho_2$ where $\rho_1 < 0 < \rho_2$, which encircles all the poles of P_1 once in the positive direction but none of the poles of P_2 . Bromwich contour L_v is a vertical contour, which goes from $\gamma_0 - i\infty$ to $\gamma_0 + i\infty$ where $\gamma_0 \in \mathbf{R}$ and leaves all the poles of P_1 to the right and all poles of P_2 to the left. Braaksma [5] showed that the contour integral (6.7) makes sense along L_h and L_v for $r \in (0, \infty)$, and two integrals along L_h and L_v coincide (see [16, Section 1.2]). Furthermore, if $\sum_{j=1}^\mu \delta_j - \sum_{i=1}^\nu \gamma_i > 0$ then $H(r)$ is an analytic function on $(0, \infty)$ and can be represented (see [16, Theorem 1.2]) as

$$H(r) = \sum_{i=1}^m \sum_{k=0}^\infty \text{Res}_{z=\hat{d}_{ik}} \left[\frac{\prod_{j=1}^m \Gamma(d_j + \delta_j z) \prod_{j=1}^n \Gamma(1 - c_j - \gamma_j z)}{\prod_{j=m+1}^\mu \Gamma(1 - d_j - \delta_j z) \prod_{j=n+1}^\nu \Gamma(c_j + \gamma_j z)} r^{-z} \right], \quad (6.9)$$

where $\hat{d}_{ik} := -(d_i + k)/\delta_i \in P_1$ are poles of the integrand in the contour integral, $i \in \{1, \dots, m\}$, and $k = 0, 1, 2, \dots$. Note that on the negative real half-axis the Mittag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0,$$

can be written as

$$E_\alpha(-x) = H_{12}^{11} \left[x \mid \begin{matrix} (0, 1) \\ (0, 1) \end{matrix} \quad (0, \alpha) \right] \quad (x > 0). \quad (6.10)$$

Indeed, by (6.9) and (6.3)

$$\begin{aligned} H_{12}^{11} \left[x \mid \begin{pmatrix} (0, 1) \\ (0, 1) \end{pmatrix} \quad (0, \alpha) \right] &= \sum_{k=0}^{\infty} \text{Res}_{z=-k} \left[\frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1-\alpha z)} x^{-z} \right] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1+k)}{k! \Gamma(\alpha k + 1)} x^k = \sum_{k=0}^{\infty} \frac{(-x)^k}{\Gamma(\alpha k + 1)} = E_{\alpha}(-x). \end{aligned}$$

We introduce some notion to introduce the asymptotic behavior of $H(r)$. Let j_s ($1 \leq j_s \leq m$) be the number such that

$$\rho_s := \frac{d_{j_s}}{\delta_{j_s}} = \min \left[\frac{d_j}{\delta_j} \right] \quad (\min \emptyset := \infty)$$

where the minimum are taken over all $\frac{d_j}{\delta_j}$ so that \hat{d}_{j_0} is a simple pole. Similarly let j_c ($1 \leq j_c \leq m$) be the number such that

$$\rho_c := \frac{d_{j_c}}{\delta_{j_c}} = \min \left[\frac{d_j}{\delta_j} \right]$$

where the minimum are taken over all $\frac{d_j}{\delta_j}$ so that \hat{d}_{j_0} is a pole with order $n_c \geq 2$. Here n_c denotes the smallest number of the orders for non-simple poles.

The following result can be proved on the basis of (6.9). See [16, Corollary 1.12.1] for the proof.

Theorem 6.1. (i) If $\rho_s < \rho_c$, then for $r \leq 1$

$$|H(r)| \leq N r^{\rho_s};$$

(ii) if $\rho_s \geq \rho_c$, then for $r \leq 1$

$$|H(r)| \leq N r^{\rho_c} |\ln r|^{n_c-1}.$$

An upper bound of $H(r)$ on $[1, \infty)$ is also well-known if $n = 0$ and $m = \mu$ in (6.7). See [16, Corollary 1.10.2] and [16, (2.2.2)].

Theorem 6.2. Suppose that $n = 0$ and $m = \mu$ in (6.7). Then for $r \geq 1$,

$$|H(r)| \leq N r^{(\Lambda+1/2)\omega^{-1}} \exp\{-\omega \eta^{-1/\omega} r^{1/\omega}\}, \quad (6.11)$$

where

$$\Lambda := \sum_{j=1}^{\mu} d_j - \sum_{i=1}^{\nu} c_i + \frac{\nu - \mu}{2}, \quad \omega := \sum_{j=1}^{\mu} \delta_j - \sum_{i=1}^{\nu} \gamma_i, \quad \eta := \prod_{j=1}^{\mu} \delta_j^{\delta_j} \prod_{i=1}^{\nu} \gamma_i^{-\gamma_i}.$$

Theorem 6.3. Suppose that $n = 0$ and $m = \mu$ in (6.7). Then

$$\frac{d}{dr} H(r) = -r^{-1} H_{\nu+1, \mu+1}^{\mu+1, 0} \left[r \mid \begin{pmatrix} (c_1, \gamma_1) & \cdots & (c_{\nu}, \gamma_{\nu}) & (0, 1) \\ (d_1, \delta_1) & \cdots & (d_{\mu}, \delta_{\mu}) & (1, 1) \end{pmatrix} \right]. \quad (6.12)$$

Now we define $p(t, x)$ so that (6.1) holds. Fix $\alpha \in (0, 2)$ and let

$$\begin{aligned} p(t, x) &:= \pi^{-\frac{d}{2}} |x|^{-d} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{pmatrix} (1, \alpha) \\ (\frac{d}{2}, 1) \end{pmatrix} \quad (1, 1) \right] \\ &= \pi^{-\frac{d}{2}} |x|^{-d} \frac{1}{2\pi i} \int_{L_v} \frac{\Gamma(\frac{d}{2} + z) \Gamma(1 + z)}{\Gamma(1 + \alpha z)} \left(\frac{1}{4} t^{-\alpha} |x|^2 \right)^{-z} dz. \end{aligned} \quad (6.13)$$

One can easily check that (6.8) is satisfied. Hence we can take the Bromwich contour $L = L_v$, which runs along from $-\gamma - i\infty$ to $-\gamma + i\infty$, i.e.

$$L_v := \{z \in \mathbb{C} : \Re[z] = -\gamma\}$$

where

$$0 < \gamma < \min\{1, \frac{d-1}{4}, \frac{1}{\alpha}\} \quad \text{if } d \geq 2 \quad \text{and} \quad 0 < \gamma < \min\{\frac{1}{2}, \frac{1}{\alpha}\} \quad \text{if } d = 1.$$

Note that

$$\min\{\rho_s, \rho_c\} = \begin{cases} \rho_s = 1 & \text{if } d \geq 3 \\ \rho_c = 1, n_c = 2 & \text{if } d = 2 \\ \rho_s = \frac{1}{2} & \text{if } d = 1. \end{cases}$$

By Theorem 6.1 and Theorem 6.2,

$$p(t, \cdot) \in L_1(\mathbf{R}^d), \quad \forall t > 0.$$

For the asymptotic behavior of the integrand in (6.13) we use Stirling's approximation. Write $z = -\gamma + i\tau$, $\tau \in (-\infty, \infty)$. Then by (6.6), for $\rho \in (0, \infty)$, $t \in (0, \infty)$, and large $|\tau|$,

$$\left| \frac{\Gamma(\frac{d}{2} + z)\Gamma(1 + z)}{\Gamma(1 + \alpha z)} \left(\frac{1}{4}t^{-\alpha}\rho^2\right)^{-z} \right| \leq N \left(\frac{1}{4}t^{-\alpha}\rho^2\right)^{\gamma} |\tau|^{c_1} e^{-c_2|\tau|}, \quad (6.14)$$

where

$$c_1 := -\gamma(2 - \alpha) + \frac{d-1}{2}, \quad c_2 := \frac{\pi}{2}(2 - \alpha).$$

Therefore,

$$\sup_{z \in L} \left| \frac{\Gamma(\frac{d}{2} + z)\Gamma(1 + z)}{\Gamma(1 + \alpha z)} \left(\frac{1}{4}t^{-\alpha}\rho^2\right)^{-z} \right| \leq N \left(\frac{1}{4}t^{-\alpha}\rho^2\right)^{\gamma}, \quad (6.15)$$

because for fixed $a \notin \{0, -1, -2, -3, \dots\}$ the mapping $b \mapsto \Gamma(a + ib)$ is a continuous function and does not vanish on \mathbf{R} . Hence

$$\begin{aligned} & \left| \int_L \frac{\Gamma(\frac{d}{2} + z)\Gamma(1 + z)}{\Gamma(1 + \alpha z)} \left(\frac{1}{4}t^{-\alpha}\rho^2\right)^{-z} dz \right| \\ & \leq \int_{-\infty}^{\infty} \left| \frac{\Gamma(\frac{d}{2} - \gamma + i\tau)\Gamma(1 - \gamma + i\tau)}{\Gamma(1 - \alpha\gamma + i\alpha\tau)} \left(\frac{1}{4}t^{-\alpha}\rho^2\right)^{\gamma - i\tau} \right| d\tau \\ & \leq N t^{-\alpha\gamma} \rho^{2\gamma} \left\{ N + \int_{|\tau| \geq 1} |\tau|^{c_1} e^{-c_2|\tau|} d\tau \right\} \leq N t^{-\alpha\gamma} \rho^{2\gamma}. \end{aligned} \quad (6.16)$$

We remark that (6.14), (6.15), and (6.16) hold for any $0 < \gamma < \min\{1, \frac{d}{2}, \frac{1}{\alpha}\}$.

Now we prove (6.1). First assume $d \geq 2$. By the formula for the Fourier transform of a radial function (see [42, Theorem IV.3.3]), we have

$$\mathcal{F}\{p(t, \cdot)\}(\xi) = \frac{2^{d/2}}{|\xi|^{\frac{d}{2}-1}} \int_0^\infty \rho^{-\frac{d}{2}} H_{12}^{20} \left[\frac{1}{4}t^{-\alpha}\rho^2 \mid \begin{matrix} (1, \alpha) \\ (\frac{d}{2}, 1) \end{matrix} \begin{matrix} (1, 1) \end{matrix} \right] J_{\frac{d}{2}-1}(|\xi|\rho) d\rho,$$

where $J_{\frac{d}{2}-1}$ is the Bessel function of the first kind of order $\frac{d}{2} - 1$, i.e. for $r \in [0, \infty)$,

$$J_{\frac{d}{2}-1}(r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \frac{d}{2})} \left(\frac{r}{2}\right)^{2k-1+d/2}.$$

It is well-known (e.g. [16, (2.6.3)]) that if $m > -1$ then

$$J_m(t) = \begin{cases} O(t^m), & t \rightarrow 0+ \\ O(t^{-1/2}), & t \rightarrow \infty. \end{cases}$$

Thus

$$\int_0^\infty |\rho^{-\frac{d}{2}+2\gamma} J_{\frac{d}{2}-1}(|\xi|\rho)| d\rho \leq \int_0^1 \rho^{2\gamma-1} d\rho + \int_1^\infty \rho^{-\frac{d}{2}+2\gamma-\frac{1}{2}} d\rho < \infty \quad (6.17)$$

since $0 < \gamma < \min\{1, \frac{d-1}{4}, \frac{1}{\alpha}\}$. By the definition of the Fox H-function,

$$\begin{aligned} & \int_0^\infty \rho^{-\frac{d}{2}} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} \rho^2 \mid \begin{matrix} (1, \alpha) \\ (\frac{d}{2}, 1) \end{matrix} \begin{matrix} (1, 1) \end{matrix} \right] J_{\frac{d}{2}-1}(|\xi|\rho) d\rho \\ &= \frac{1}{2\pi i} \int_0^\infty \rho^{-\frac{d}{2}} \left[\int_L \frac{\Gamma(\frac{d}{2}+z)\Gamma(1+z)}{\Gamma(1+\alpha z)} \left(\frac{1}{4} t^{-\alpha} \rho^2 \right)^{-z} dz \right] J_{\frac{d}{2}-1}(|\xi|\rho) d\rho. \end{aligned} \quad (6.18)$$

Combining (6.16) and (6.17), we have

$$\int_0^\infty \int_L \left| \rho^{-\frac{d}{2}} \frac{\Gamma(\frac{d}{2}+z)\Gamma(1+z)}{\Gamma(1+\alpha z)} \left(\frac{1}{4} t^{-\alpha} \rho^2 \right)^{-z} J_{\frac{d}{2}-1}(|\xi|\rho) \right| |dz| d\rho < \infty.$$

Thus we can apply Fubini's theorem to (6.18). Furthermore, since

$$-\frac{d}{2} < \Re[-\frac{d}{2} - 2z] < -\frac{1}{2} \quad \forall z \in L,$$

by using the formula [16, (2.6.4)]

$$\int_0^\infty \rho^{-\frac{d}{2}-2z} J_{\frac{d}{2}-1}(|\xi|\rho) d\rho = 2^{-\frac{d}{2}-2z} |\xi|^{\frac{d}{2}-1+2z} \frac{\Gamma(-z)}{\Gamma(\frac{d}{2}+z)},$$

we get

$$\begin{aligned} & \int_0^\infty \rho^{-\frac{d}{2}} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} \rho^2 \mid \begin{matrix} (1, \alpha) \\ (\frac{d}{2}, 1) \end{matrix} \begin{matrix} (1, 1) \end{matrix} \right] J_{\frac{d}{2}-1}(|\xi|\rho) d\rho \\ &= \frac{1}{2\pi i} \int_L \left[\int_0^\infty \rho^{-\frac{d}{2}-2z} J_{\frac{d}{2}-1}(|\xi|\rho) d\rho \right] \frac{\Gamma(\frac{d}{2}+z)\Gamma(1+z)}{\Gamma(1+\alpha z)} \left(\frac{1}{4} t^{-\alpha} \right)^{-z} dz \\ &= \frac{|\xi|^{\frac{d}{2}-1}}{2^{d/2}} \frac{1}{2\pi i} \int_L \frac{\Gamma(\frac{d}{2}+z)\Gamma(1+z)\Gamma(-z)}{\Gamma(1+\alpha z)\Gamma(\frac{d}{2}+z)} (t^{-\alpha} |\xi|^{-2})^{-z} dz \\ &= \frac{|\xi|^{\frac{d}{2}-1}}{2^{d/2}} \frac{1}{2\pi i} \int_{-L} \frac{\Gamma(1-z)\Gamma(z)}{\Gamma(1-\alpha z)} (t^\alpha |\xi|^2)^{-z} dz \\ &= \frac{|\xi|^{\frac{d}{2}-1}}{2^{d/2}} H_{12}^{11} \left[t^\alpha |\xi|^2 \mid \begin{matrix} (0, 1) \\ (0, 1) \end{matrix} \begin{matrix} (0, \alpha) \end{matrix} \right] = \frac{|\xi|^{\frac{d}{2}-1}}{2^{d/2}} E_\alpha(-t^\alpha |\xi|^2) \end{aligned}$$

where the last equality is due to (6.10). Therefore (6.1) is proved for $d \geq 2$.

Next let $d = 1$. By (3.10),

$$\mathcal{L} [E_\alpha(-t^\alpha |\xi|^2)](s) = \int_0^\infty e^{-st} E_\alpha(-t^\alpha |\xi|^2) dt = \frac{s^{\alpha-1}}{s^\alpha + |\xi|^2}.$$

Thus it suffices to prove

$$\mathcal{L} [\mathcal{F} \{p(t, \cdot)\}](s) = \int_0^\infty e^{-st} \mathcal{F} \{p(t, \cdot)\}(\xi) dt = \frac{s^{\alpha-1}}{s^\alpha + |\xi|^2}. \quad (6.19)$$

By Theorems 6.1 and 6.2 it holds that for each $s > 0$,

$$\int_0^\infty \int_{\mathbf{R}} e^{-st} |p(t, x)| dx dt < \infty.$$

Hence by Fubini's theorem, $\mathcal{L}[\mathcal{F}\{p(t, \cdot)\}](s) = \mathcal{F}\{\mathcal{L}[p(\cdot, x)](s)\}$. Due to (6.4) and (6.16),

$$\begin{aligned} \mathcal{L}[p(\cdot, x)](s) &= \int_0^\infty e^{-st} p(t, x) dt \\ &= \pi^{-\frac{1}{2}} |x|^{-1} \int_0^\infty e^{-st} \left[\frac{1}{2\pi i} \int_L \frac{\Gamma(\frac{1}{2} + z) \Gamma(1 + z)}{\Gamma(1 + \alpha z)} \left(\frac{1}{4} t^{-\alpha} |x|^2 \right)^{-z} dz \right] dt \\ &= \frac{\pi^{-\frac{1}{2}} |x|^{-1}}{2\pi i} \int_L \frac{\Gamma(\frac{1}{2} + z) \Gamma(1 + z)}{\Gamma(1 + \alpha z)} \left(\frac{1}{4} |x|^2 \right)^{-z} \left[\int_0^\infty e^{-st} t^{\alpha z} dt \right] dz \\ &= \frac{\pi^{-\frac{1}{2}} |x|^{-1} s^{-1}}{2\pi i} \int_L \Gamma\left(\frac{1}{2} + z\right) \Gamma(1 + z) \left(\frac{1}{4} s^\alpha |x|^2 \right)^{-z} dz. \end{aligned}$$

Furthermore by [16, (2.9.19)],

$$\begin{aligned} &\frac{1}{2\pi i} \int_L \Gamma\left(\frac{1}{2} + z\right) \Gamma(1 + z) \left(\frac{1}{4} s^\alpha |x|^2 \right)^{-z} dz \\ &= H_{02}^{20} \left[\frac{1}{4} s^\alpha |x|^2 \mid \begin{matrix} (1, \alpha) \\ (\frac{d}{2}, 1) \end{matrix} \begin{matrix} (1, 1) \end{matrix} \right] = 2 \left(\frac{s^{\alpha/2} |x|}{2} \right)^{3/2} K_{1/2}(s^{\alpha/2} |x|), \end{aligned}$$

where $K_\eta(z)$ is called the modified Bessel function of the second kind¹ or the MacDonald function and it satisfies (see [1, 9.7.2])

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2}} z^{-1/2} e^{-z}, \quad |\arg z| < \frac{3\pi}{2}.$$

Hence we obtain

$$\mathcal{L}[p(\cdot, x)](s) = \frac{s^{\alpha/2-1}}{2} \exp\{-s^{\alpha/2} |x|\},$$

which obviously implies that

$$\begin{aligned} \mathcal{F}\{\mathcal{L}[p(\cdot, x)](s)\}(\xi) &= \frac{s^{\alpha/2-1}}{2} \int_{-\infty}^\infty e^{-ix\xi} \exp\{-s^{\alpha/2} |x|\} dx \\ &= \frac{s^{\alpha/2-1}}{2} \cdot \frac{2s^{\alpha/2}}{s^\alpha + |\xi|^2} = \frac{s^{\alpha-1}}{s^\alpha + |\xi|^2}. \end{aligned}$$

Therefore (6.19) is proved, and (6.1) holds.

6.2. Representation of $q(t, x)$ and $K(t, x)$. Let $\alpha \in (0, 2)$ and recall

$$p(t, x) := \pi^{-\frac{d}{2}} |x|^{-d} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{matrix} (1, \alpha) \\ (\frac{d}{2}, 1) \end{matrix} \begin{matrix} (1, 1) \end{matrix} \right].$$

By Theorems 6.2 and (6.12), if $t \neq 0$ and $x \neq 0$ then $p(t, x)$ is differentiable in t and $\lim_{t \rightarrow 0+} p(t, x) = 0$. Thus we can define

$$q(t, x) := \begin{cases} I_t^{\alpha-1} p(t, x), & \alpha \in (1, 2) \\ D_t^{1-\alpha} p(t, x), & \alpha \in (0, 1) \end{cases}, \quad K(t, x) := \frac{\partial}{\partial t} p(t, x).$$

¹It has also been called the modified Bessel function of the third kind.

In this subsection, we derive the following representations:

$$q(t, x) = \pi^{-\frac{d}{2}} |x|^{-d} t^{\alpha-1} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{matrix} (\alpha, \alpha) \\ (\frac{d}{2}, 1) \end{matrix} \quad (1, 1) \right] \quad (6.20)$$

and

$$K(t, x) = \pi^{-\frac{d}{2}} |x|^{-d} t^{-1} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{matrix} (0, \alpha) \\ (\frac{d}{2}, 1) \end{matrix} \quad (1, 1) \right]. \quad (6.21)$$

We consider the Bromwich contour

$$L_v := \{z \in \mathbb{C} : \Re[z] = -\gamma\}, \quad 0 < \gamma < \min\{1, \frac{d}{2}, \frac{1}{\alpha}\}.$$

Let $\beta > 0$. By (6.16),

$$\begin{aligned} & \int_0^t (t-s)^{\beta-1} \left(\int_L \left| \frac{\Gamma(\frac{d}{2} + z) \Gamma(1+z)}{\Gamma(1+\alpha z)} \left(\frac{1}{4} s^{-\alpha} |x|^2 \right)^{-z} \right| |dz| \right) ds \\ & \leq N |x|^{2\gamma} \int_0^t (t-s)^{\beta-1} (t-s)^{-\alpha\gamma} ds < \infty. \end{aligned}$$

Thus by Fubini's theorem and (6.4),

$$\begin{aligned} I_t^\beta p(t, x) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} p(s, x) ds \\ &= \frac{\pi^{-\frac{d}{2}} |x|^{-d}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[\frac{1}{2\pi i} \int_L \frac{\Gamma(\frac{d}{2} + z) \Gamma(1+z)}{\Gamma(1+\alpha z)} \left(\frac{1}{4} s^{-\alpha} |x|^2 \right)^{-z} dz \right] ds \\ &= \frac{\pi^{-\frac{d}{2}} |x|^{-d}}{2\pi i} \int_L \frac{\Gamma(\frac{d}{2} + z) \Gamma(1+z)}{\Gamma(1+\alpha z)} \left[\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\alpha z} ds \right] \left(\frac{1}{4} |x|^2 \right)^{-z} dz \\ &= \pi^{-\frac{d}{2}} |x|^{-d} t^\beta \frac{1}{2\pi i} \int_L \frac{\Gamma(\frac{d}{2} + z) \Gamma(1+z)}{\Gamma(1+\beta + \alpha z)} \left(\frac{1}{4} t^{-\alpha} |x|^2 \right)^{-z} dz \\ &= \pi^{-\frac{d}{2}} |x|^{-d} t^\beta H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{matrix} (1+\beta, \alpha) \\ (\frac{d}{2}, 1) \end{matrix} \quad (1, 1) \right]. \end{aligned} \quad (6.22)$$

Therefore, (6.20) is proved for $\alpha \in (1, 2)$.

Next we differentiate kernels with respect to t . Write $z = -\gamma + i\tau$. Following the method used to prove (6.14) and (6.15), for any $\alpha \in (0, 2)$ and $\beta \in [0, 2)$ we have

$$\begin{aligned} & \left| \frac{\Gamma(\frac{d}{2} + z) \Gamma(1+z)}{\Gamma(1+\beta + \alpha z)} \left(\frac{1}{4} |x|^2 \right)^{-z} \right| \left| \frac{d}{dt} t^{\alpha z} \right| \\ & \leq N(d, \gamma, \beta) t^{-1} \left(\frac{1}{4} t^{-\alpha} |x|^2 \right)^\gamma \left[(\gamma^2 + |\tau|^2)^{1/2} (1_{|\tau| \geq 1} |\tau|^{c_1 - \beta} e^{-c_2 |\tau|} + 1_{|\tau| \leq 1}) \right]. \end{aligned} \quad (6.23)$$

Hence the time derivative of the integrand in (6.22) is integrable in z uniformly in a neighborhood of $t > 0$, and thus by the dominated convergence theorem,

$$\begin{aligned} & \frac{d}{dt} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{array}{c} (1 + \alpha, \alpha) \\ (\frac{d}{2}, 1) \end{array} \quad (1, 1) \right] \\ &= \frac{1}{2\pi i} \frac{d}{dt} \left\{ \int_L \frac{\Gamma(\frac{d}{2} + z) \Gamma(1 + z)}{\Gamma(1 + \alpha + \alpha z)} \left(\frac{1}{4} t^{-\alpha} |x|^2 \right)^{-z} dz \right\} \\ &= \frac{t^{-1}}{2\pi i} \int_L \frac{\Gamma(\frac{d}{2} + z) \Gamma(1 + z)}{\Gamma(1 + \alpha + \alpha z)} \alpha z \left(\frac{1}{4} t^{-\alpha} |x|^2 \right)^{-z} dz. \end{aligned}$$

Using the relation

$$\frac{\alpha z}{\Gamma(\alpha + \alpha z + 1)} = \frac{1}{\Gamma(\alpha + \alpha z)} - \frac{\alpha}{\Gamma(\alpha + \alpha z + 1)},$$

we obtain

$$\begin{aligned} & \frac{d}{dt} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{array}{c} (1 + \alpha, \alpha) \\ (\frac{d}{2}, 1) \end{array} \quad (1, 1) \right] \\ &= \frac{t^{-1}}{2\pi i} \int_L \left\{ \frac{\Gamma(\frac{d}{2} + z) \Gamma(1 + z)}{\Gamma(\alpha + \alpha z)} - \alpha \frac{\Gamma(\frac{d}{2} + z) \Gamma(1 + z)}{\Gamma(\alpha + \alpha z + 1)} \right\} \left(\frac{1}{4} t^{-\alpha} |x|^2 \right)^{-z} dz \\ &= t^{-1} \left\{ H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{array}{c} (\alpha, \alpha) \\ (\frac{d}{2}, 1) \end{array} \quad (1, 1) \right] - \alpha H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{array}{c} (1 + \alpha, \alpha) \\ (\frac{d}{2}, 1) \end{array} \quad (1, 1) \right] \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} D_t^{1-\alpha} p(t, x) &= \frac{d}{dt} I_t^\alpha p(t, x) \\ &= \frac{d}{dt} \left\{ \pi^{-\frac{d}{2}} |x|^{-d} t^\alpha H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{array}{c} (1 + \alpha, \alpha) \\ (\frac{d}{2}, 1) \end{array} \quad (1, 1) \right] \right\} \\ &= \pi^{-d/2} |x|^{-d} \left\{ \alpha t^{\alpha-1} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{array}{c} (1 + \alpha, \alpha) \\ (\frac{d}{2}, 1) \end{array} \quad (1, 1) \right] \right. \\ &\quad \left. + t^\alpha \frac{d}{dt} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{array}{c} (1 + \alpha, \alpha) \\ (\frac{d}{2}, 1) \end{array} \quad (1, 1) \right] \right\} \\ &= \pi^{-d/2} |x|^{-d} t^{\alpha-1} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{array}{c} (\alpha, \alpha) \\ (\frac{d}{2}, 1) \end{array} \quad (1, 1) \right]. \end{aligned}$$

Similarly, using the relation

$$\frac{\alpha z}{\Gamma(\alpha z + 1)} = \frac{1}{\Gamma(\alpha z)},$$

we have

$$\begin{aligned} K(t, x) &= \frac{d}{dt} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{array}{c} (1, \alpha) \\ (\frac{d}{2}, 1) \end{array} \quad (1, 1) \right] \\ &= \pi^{-\frac{d}{2}} |x|^{-d} t^{-1} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x|^2 \mid \begin{array}{c} (0, \alpha) \\ (\frac{d}{2}, 1) \end{array} \quad (1, 1) \right]. \end{aligned}$$

Therefore (6.20) and (6.21) are proved.

6.3. Estimates of $p(t, x)$ and $q(t, x)$. In this subsection we prove Lemma 3.1(ii) and Lemma 3.2. Since the case $\alpha = 1$ is easier, we assume $\alpha \neq 1$.

By (6.7) and (6.9) with $n = 0$ and $m = \mu$,

$$\begin{aligned} H_{\nu\mu}^{\mu 0} \left[r \mid \begin{array}{ccc} (c_1, \gamma_1) & \cdots & (c_\nu, \gamma_\nu) \\ (d_1, \delta_1) & \cdots & (d_\mu, \delta_\mu) \end{array} \right] &= \sum_{i=1}^m \sum_{k=0}^{\infty} \text{Res}_{z=\hat{d}_{ik}} \left[\frac{\prod_{j=1}^{\mu} \Gamma(d_j + \delta_j z)}{\prod_{j=1}^{\nu} \Gamma(c_j + \gamma_j z)} r^{-z} \right] \\ &= \sum_{i=1}^m \sum_{k=0}^{\infty} \lim_{z \rightarrow \hat{d}_{ik}} \left\{ \left(\frac{d}{dz} \right)^{n_{ik}-1} \left(\frac{(z - \hat{d}_{ik})^{n_{ik}}}{(n_{ik} - 1)!} \frac{\prod_{j=1}^{\mu} \Gamma(d_j + \delta_j z)}{\prod_{j=1}^{\nu} \Gamma(c_j + \gamma_j z)} r^{-z} \right) \right\}, \end{aligned}$$

where $\hat{d}_{ik} = -(d_i + k)/\delta_i \in P_1$ is a pole of the integrand in the contour integral and n_{ik} is its order for $i = 1, \dots, \mu$ and $k = 0, 1, 2, \dots$.

Let $R := t^{-\alpha}|x|^2$, and denote

$$H_{k,l}^p(R) := H_{1+k+l}^{2+k+l} \begin{array}{c} 0 \\ 2+k+l \end{array} \left[\frac{1}{4}R \mid \begin{array}{cccc} (1, \alpha) & (0, 1) & \cdots & (0, 1) \\ (\frac{d}{2}, 1) & (1, 1) & \cdots & (1, 1) \end{array} \right]$$

and

$$H_{k,l}^q(R) := H_{1+k+l}^{2+k+l} \begin{array}{c} 0 \\ 2+k+l \end{array} \left[\frac{1}{4}R \mid \begin{array}{cccc} (\alpha, \alpha) & (0, 1) & \cdots & (0, 1) \\ (\frac{d}{2}, 1) & (1, 1) & \cdots & (1, 1) \end{array} \right],$$

where $k, l = 0, 1, 2, \dots$. Then by (6.12),

$$\begin{aligned} \left| D_{x^i} \left(|x|^{-d} H_{k,0}^p(R) \right) \right| &= \left| -dx^i |x|^{-d-2} H_{k,0}^p(R) - 2x^i |x|^{-d-2} H_{k,1}^p(R) \right| \\ &= \left| -x^i |x|^{-d-2} [dH_{k,0}^p(R) + 2H_{k,1}^p(R)] \right| \\ &\leq N |x|^{-d-1} |dH_{k,0}^p(R) + 2H_{k,1}^p(R)|, \end{aligned}$$

and

$$\begin{aligned} &\left| D_{x^j} D_{x^i} \left(|x|^{-d} H_{k,0}^p(R) \right) \right| \\ &= \left| D_{x^j} \left(-x^i |x|^{-d-2} [dH_{k,0}^p(R) + 2H_{k,1}^p(R)] \right) \right| \\ &= \left| (-\delta_{ij} |x|^{-d-2} + (d+2)x^i x^j |x|^{-d-4}) [dH_{k,0}^p(R) + 2H_{k,1}^p(R)] \right. \\ &\quad \left. + 2x^i x^j |x|^{-d-4} [dH_{k,1}^p(R) + 2H_{k,2}^p(R)] \right| \\ &\leq N |x|^{-d-2} \sum_{l=1}^2 |dH_{k,l-1}^p(R) + 2H_{k,l}^p(R)|. \end{aligned}$$

Inductively, for any $m = 3, 4, \dots$ and $k = 0, 1, \dots$, we have

$$\left| D_x^m \left(|x|^{-d} H_{k,0}^p(R) \right) \right| \leq N |x|^{-d-m} \sum_{l=1}^m |dH_{k,l-1}^p(R) + 2H_{k,l}^p(R)|.$$

Hence

$$\begin{aligned}
|\partial_t^n D_x^m p(t, x)| &= \left| D_x^m \left(|x|^{-d} \partial_t^n H_{12}^{20} \left[\frac{1}{4} R \mid \begin{pmatrix} 1, \alpha \\ \frac{d}{2}, 1 \end{pmatrix} \quad (1, 1) \right] \right) \right| \\
&\leq N t^{-n} \sum_{k=1}^n \left| D_x^m \left(|x|^{-d} H_{k,0}^p(R) \right) \right| \\
&\leq N |x|^{-d-m} t^{-n} \sum_{k=1}^n \sum_{l=1}^m \left| dH_{k,l-1}^p(R) + 2H_{k,l}^p(R) \right|. \tag{6.24}
\end{aligned}$$

Similarly,

$$\left| D_x^m \left(|x|^{-d} H_{k,0}^q(R) \right) \right| \leq N |x|^{-d-m} \sum_{l=1}^m \left| dH_{k,l-1}^q(R) + 2H_{k,l}^q(R) \right|$$

and

$$\begin{aligned}
|\partial_t^n D_x^m q(t, x)| &\leq N t^{-n+\alpha-1} \sum_{k=1}^n \left| D_x^m \left(|x|^{-d} H_{k,0}^q(R) \right) \right| \\
&\leq N |x|^{-d-m} t^{-n+\alpha-1} \sum_{k=0}^n \sum_{l=1}^m \left| dH_{k,l-1}^q(R) + 2H_{k,l}^q(R) \right|. \tag{6.25}
\end{aligned}$$

Now we prove (3.2) and (3.4). If $R \geq 1$, by Theorem 6.2

$$H_{k,l}^p(R) \leq N R^{(\frac{d}{2}+k+l)/(2-\alpha)} \exp\{-\sigma R^{\frac{1}{2-\alpha}}\}$$

and

$$H_{k,l}^q(R) \leq N R^{(\frac{d}{2}+k+l+1-\alpha)/(2-\alpha)} \exp\{-\sigma R^{\frac{1}{2-\alpha}}\},$$

where $\sigma = (2-\alpha)\alpha^{\alpha/(2-\alpha)}$. Hence by (6.24)

$$\begin{aligned}
|\partial_t^n D_x^m p(t, x)| &\leq N |x|^{-d-m} t^{-n} \sum_{k=1}^n \sum_{l=0}^m R^{(\frac{d}{2}+k+l)/(2-\alpha)} \exp\{-\sigma R^{\frac{1}{2-\alpha}}\} \\
&\leq N (t^{-\alpha/2})^{d+m} t^{-n} \exp\{-(\sigma/2) t^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{2}{2-\alpha}}\} \\
&\leq N t^{\frac{-\alpha(d+m)}{2}-n} \exp\{-(\sigma/2) t^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{2}{2-\alpha}}\}.
\end{aligned}$$

Similarly, by (6.25)

$$|\partial_t^n D_x^m q(t, x)| \leq N t^{\frac{-\alpha(d+m)}{2}-n+\alpha-1} \exp\{-(\sigma/2) t^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{2}{2-\alpha}}\}.$$

Thus (3.2) and (3.4) are proved.

To prove (3.3), we recall (6.2). For $k, l = 0, 1, \dots$, denote

$$\Theta_{k,l}^p(z) = \frac{\Gamma(\frac{d}{2}+z)\{\Gamma(1+z)\}^{k+l+1}}{\Gamma(1+\alpha z)\{\Gamma(z)\}^{k+l}} = \frac{\Gamma(\frac{d}{2}+z)\Gamma(1+z)}{\Gamma(1+\alpha z)} z^{k+l}.$$

First we assume that d is an odd number. Then $\Theta_{k,l}^p$ has simple poles at $d_{1j} = -1-j$ and $d_{2j} = -\frac{d}{2} - j$ for $j = 0, 1, 2, \dots$. Due to (6.3) and (6.9), for $R \leq 1$ we have

$$\begin{aligned} H_{k,l}^p(R) &= \sum_{i=1}^2 \sum_{j=0}^{\infty} \text{Res}_{z=d_{ij}} \left[\Theta_{k,l}^p R^{-z} \right] \\ &= \sum_{i=1}^2 \sum_{j=0}^{\infty} \lim_{z \rightarrow d_{ij}} \left((z - d_{ij}) \Theta_{k,l}^p(z) R^{-z} \right) \\ &= \sum_{j=0}^{\infty} (-1-j)^{k+l} \cdot \frac{(-1)^j}{j!} \cdot \frac{\Gamma(\frac{d}{2} - 1 - j)}{\Gamma(1 - \alpha - j\alpha)} R^{1+j} \\ &\quad + \sum_{j=0}^{\infty} \left(-\frac{d}{2} - j \right)^{k+l} \cdot \frac{(-1)^j}{j!} \cdot \frac{\Gamma(1 - \frac{d}{2} - j)}{\Gamma(1 - \frac{\alpha d}{2} - \alpha j)} R^{\frac{d}{2}+j} \\ &= (-1)^{k+l} \frac{\Gamma(\frac{d}{2} - 1)}{\Gamma(1 - \alpha)} R + \left(-\frac{d}{2} \right)^{k+l} \frac{\Gamma(1 - \frac{d}{2})}{\Gamma(1 - \frac{\alpha d}{2})} R^{\frac{d}{2}} + O(R)(R^{\frac{d}{2}+1} + R^2), \end{aligned}$$

where $O(R)$ is bounded in $(0, 1]$ by (6.5) and (6.2). Hence

$$|H_{k,0}^p(R)| \leq N(R + R^{1/2} \cdot 1_{d=1}). \quad (6.26)$$

Moreover since

$$d \left(-\frac{d}{2} \right)^{k+l-1} + 2 \left(-\frac{d}{2} \right)^{k+l} = 0 \quad l = 1, 2, \dots, \quad (6.27)$$

it holds that for all $l \geq 1$

$$|dH_{k,l-1}^p(R) + 2H_{k,l}^p(R)| \leq NR. \quad (6.28)$$

If d is an even number, $\Theta_{k,l}^p$ has a simple pole at $d_{1j} = -1-j$ for $0 \leq j \leq \frac{d}{2} - 2$ and a pole of order 2 at $d_{2j} = -\frac{d}{2} - j$ for $j = 0, 1, 2, \dots$. Hence by (6.3) and (6.9), for $R \leq 1$ we have

$$\begin{aligned} H_{k,l}^p(R) &= \sum_{j=0}^{\frac{d}{2}-2} \text{Res}_{z=d_{1j}} \left[\Theta_{k,l}^p R^{-z} \right] 1_{d \neq 2} + \sum_{j=0}^{\infty} \text{Res}_{z=d_{2j}} \left[\Theta_{k,l}^p R^{-z} \right] \\ &= \sum_{j=0}^{\frac{d}{2}-2} \lim_{z \rightarrow d_{1j}} \left((z - d_{1j}) \Theta_{k,l}^p(z) R^{-z} \right) 1_{d \neq 2} + \sum_{j=0}^{\infty} \lim_{z \rightarrow d_{2j}} \frac{d}{dz} \left((z - d_{2j})^2 \Theta_{k,l}^p(z) R^{-z} \right) \\ &= \sum_{j=0}^{\frac{d}{2}-2} (-1-j)^{k+l} \cdot \frac{(-1)^j}{j!} \cdot \frac{\Gamma(\frac{d}{2} - 1 - j)}{\Gamma(1 - \alpha(1+j))} R^{1+j} \cdot 1_{d \neq 2} \\ &\quad - \sum_{j=0}^{\infty} \left(-\frac{d}{2} - j \right)^{k+l} \cdot \frac{(-1)^{\frac{d}{2}+2j-1}}{j! (\frac{d}{2} + j - 1)!} \cdot \frac{R^{\frac{d}{2}+j} \ln R}{\Gamma(1 - \frac{\alpha d}{2} - j\alpha)} \\ &\quad + \sum_{j=0}^{\infty} \text{Res}_{z=d_{2j}} \left[\Theta_{k,l}^p(z) \right] R^{\frac{d}{2}+j} \\ &= (-1)^{k+l} \frac{\Gamma(\frac{d}{2} - 1)}{\Gamma(1 - \alpha)} R \cdot 1_{d \neq 2} - \left(-\frac{d}{2} \right)^{k+l} \cdot \frac{(-1)^{\frac{d}{2}-1}}{(\frac{d}{2} - 1)!} \cdot \frac{R^{\frac{d}{2}} \ln R}{\Gamma(1 - \frac{\alpha d}{2})} + O(R)(R^{\frac{d}{2}} + R^2), \end{aligned}$$

where $O(R)$ is bounded in $(0, 1]$ due to (6.5) and (6.2). Hence

$$|H_{k,0}^p(R)| \leq (R + R \ln R \cdot 1_{d=2}). \quad (6.29)$$

Moreover by (6.27) again, if $l \geq 1$ then

$$|dH_{k,l-1}^p(R) + 2H_{k,l}^p(R)| \leq NR. \quad (6.30)$$

Therefore due to (6.24), (6.26), (6.28), (6.29), and (6.30), we obtain for any $R \leq 1$

$$|\partial_t^n p(t, x)| \leq N|x|^{-d}t^{-n}(R + R^{1/2} \cdot 1_{d=1} + R \ln R \cdot 1_{d=2}),$$

and for any $R \leq 1$ and $m \in \mathbb{N}$

$$|\partial_t^n D_x^m p(t, x)| \leq N|x|^{-d-m}t^{-n}R,$$

where N depends only on d, m, n , and α . Therefore Lemma 3.1(ii) is proved.

Next we prove (3.5). For $k, l = 0, 1, \dots$, denote

$$\Theta_{k,l}^q(z) = \frac{\Gamma(\frac{d}{2} + z)\{\Gamma(1+z)\}^{k+l+1}}{\Gamma(\alpha + \alpha z)\{\Gamma(z)\}^{k+l}} = \frac{\Gamma(\frac{d}{2} + z)\Gamma(1+z)}{\Gamma(\alpha(1+z))}z^{k+l}.$$

We remark that $\Theta_{k,l}^q(z)$ does not have a pole at $z = -1$ unless $d = 2$. First we assume that d is an odd number. Then $\Theta_{k,l}^q$ has a simple pole at $d_{1j} = -2 - j$ and $d_{2j} = -\frac{d}{2} - j$ for $j = 0, 1, 2, \dots$. By (6.3) and (6.9), for $R \leq 1$ we have

$$\begin{aligned} H_{k,l}^q(R) &= \sum_{i=1}^2 \sum_{j=0}^{\infty} \text{Res}_{z=d_{ij}} \left[\Theta_{k,l}^q R^{-z} \right] = \sum_{i=1}^2 \sum_{j=0}^{\infty} \lim_{z \rightarrow d_{ij}} \left((z - d_{ij}) \Theta_{k,l}^q(z) R^{-z} \right) \\ &= \sum_{j=0}^{\infty} (-2-j)^{k+l} \cdot \frac{(-1)^{1+j}}{(1+j)!} \cdot \frac{\Gamma(\frac{d}{2} - 2 - j)}{\Gamma(-\alpha(1+j))} R^{2+j} \\ &\quad + \sum_{j=0}^{\infty} \left(-\frac{d}{2} - j \right)^{k+l} \cdot \frac{(-1)^j}{j!} \cdot \frac{\Gamma(1 - \frac{d}{2} - j)}{\Gamma(\alpha - \frac{\alpha d}{2} - j\alpha)} R^{\frac{d}{2}+j} \\ &= -(-2)^{k+l} \frac{\Gamma(\frac{d}{2} - 2)}{\Gamma(-\alpha)} R^2 + \left(-\frac{d}{2} \right)^{k+l} \frac{\Gamma(1 - \frac{d}{2})}{\Gamma(\alpha - \frac{\alpha d}{2})} R^{\frac{d}{2}} + O(R)(R^{\frac{d}{2}+1} + R^3), \quad (6.31) \end{aligned}$$

where $O(R)$ is bounded in $(0, 1]$ as before. Hence

$$|H_{k,0}^q(R)| \leq N(R^2 + R^{d/2} \cdot 1_{d=1,3}). \quad (6.32)$$

Furthermore, by (6.27) and (6.31) if $l \geq 1$ then

$$|dH_{k,l-1}^q(R) + 2H_{k,l}^q(R)| \leq N(R^2 + R^{3/2} \cdot 1_{d=1}). \quad (6.33)$$

On the other hand, if d is even and greater than 2 then $\Theta_{k,l}^q$ has a simple pole at $d_{1j} = -1 - j$ for $1 \leq j \leq \frac{d}{2} - 2$ and a pole of order 2 at $d_{2j} = -\frac{d}{2} - j$ for $j = 0, 1, 2, \dots$. If $d = 2$, then $\Theta_{k,l}^q$ has a simple pole at -1 and a pole of order 2 at

$d_{2j} = -1 - j$ for $j = 1, 2, \dots$. Thus by (6.3) and (6.9), for $R \leq 1$ we have

$$\begin{aligned} H_{k,l}^q(R) &= \sum_{j=1}^{\frac{d}{2}-2} \text{Res}_{z=d_{1j}} \left[\Theta_{k,l}^q R^{-z} \right] 1_{d \geq 6} + \text{Res}_{z=-1} \left[\Theta_{k,l}^q R^{-z} \right] 1_{d=2} \\ &\quad + \text{Res}_{z=d_{20}} \left[\Theta_{k,l}^q R^{-z} \right] \cdot 1_{d \geq 4} + \sum_{j=1}^{\infty} \text{Res}_{z=d_{2j}} \left[\Theta_{k,l}^q R^{-z} \right] \\ &= \sum_{j=1}^{\frac{d}{2}-2} \lim_{z \rightarrow d_{1j}} \left((z - d_{1j}) \Theta_{k,l}^q(z) R^{-z} \right) 1_{d \geq 6} + (-1)^{k+l} R \cdot 1_{d=2} \\ &\quad + \lim_{z \rightarrow d_{20}} \frac{d}{dz} \left((z - d_{20})^2 \Theta_{k,l}^q(z) R^{-z} \right) 1_{d \geq 4} + \sum_{j=1}^{\infty} \lim_{z \rightarrow d_{2j}} \frac{d}{dz} \left((z - d_{2j})^2 \Theta_{k,l}^q(z) R^{-z} \right). \end{aligned}$$

By the product rule of the differentiation, the above term equals

$$\begin{aligned} &\sum_{j=1}^{\frac{d}{2}-2} (-1-j)^{k+l} \cdot \frac{(-1)^j}{j!} \cdot \frac{\Gamma(\frac{d}{2}-1-j)}{\Gamma(-\alpha j)} R^{1+j} \cdot 1_{d \geq 6} \\ &\quad + (-1)^{k+l} R \cdot 1_{d=2} + \text{Res}_{z=d_{20}} \left[\Theta_{k,l}^q(z) \right] R^{\frac{d}{2}} \cdot 1_{d \geq 4} \\ &\quad - \left(-\frac{d}{2} \right)^{k+l} \cdot \frac{(-1)^{\frac{d}{2}-1}}{(\frac{d}{2}-1)!} \cdot \frac{R^{\frac{d}{2}} \ln R}{\Gamma(\alpha - \frac{\alpha d}{2})} 1_{d \geq 4} + \sum_{j=1}^{\infty} \text{Res}_{z=d_{2j}} \left[\Theta_{k,l}^q(z) \right] R^{\frac{d}{2}+j} \\ &\quad - \sum_{j=1}^{\infty} \left(-\frac{d}{2} - j \right)^{k+l} \cdot \frac{(-1)^{\frac{d}{2}+2j-1}}{j!(\frac{d}{2}+j-1)!} \cdot \frac{R^{\frac{d}{2}+j} \ln R}{\Gamma(\alpha - \frac{\alpha d}{2} - \alpha j)} \\ &= -(-2)^{k+l} \cdot \frac{\Gamma(\frac{d}{2}-2)}{\Gamma(-\alpha)} R^2 \cdot 1_{d \geq 6} + (-1)^{k+l} R \cdot 1_{d=2} \\ &\quad + \text{Res}_{z=d_{20}} \left[\Theta_{k,l}^q(z) \right] R^{\frac{d}{2}} \cdot 1_{d \geq 4} - \left(-\frac{d}{2} \right)^{k+l} \cdot \frac{(-1)^{\frac{d}{2}-1}}{(\frac{d}{2}-1)!} \cdot \frac{R^{\frac{d}{2}} \ln R}{\Gamma(\alpha - \frac{\alpha d}{2})} 1_{d \geq 4} \\ &\quad + O(R) (R^3 + R^{\frac{d}{2}+1} (1 + |\ln R|)), \end{aligned}$$

where $O(R)$ is again bounded in $(0, 1]$. Hence

$$|H_{k,0}^q(R)| \leq N(R^2 + R \cdot 1_{d=2} + R^2 \ln R \cdot 1_{d=4}) \quad (6.34)$$

and by (6.27)

$$|dH_{k,l-1}^q(R) + 2H_{k,l}^q(R)| \leq N(R^2 + R^2 \ln R \cdot 1_{d=2}). \quad (6.35)$$

Combining (6.25), (6.32), (6.33), (6.34), and (6.35), for any $R \leq 1$ and nonnegative integer m we have

$$\begin{aligned} &|\partial_t^n D_x^m q(t, x)| \\ &\leq N|x|^{-d-m} t^{-n+\alpha-1} (R^2 + R^2 \ln R \cdot 1_{d=2}) \\ &\quad + N|x|^{-d} t^{-n+\alpha-1} \left[R^{1/2} \cdot 1_{d=1} + R \cdot 1_{d=2} + R^2 \ln R \cdot 1_{d=4} \right] 1_{m=0}, \end{aligned}$$

where N depends only on d, m, n , and α . Therefore Lemma 3.2(i) is proved.

Finally we prove Lemma 3.2(ii), that is for any $t \neq 0$ and $x \neq 0$,

$$\partial_t^\alpha p = \Delta p, \quad \frac{\partial p}{\partial t} = \Delta q. \quad (6.36)$$

To prove the first assertion above one may try to show that their Fourier transforms coincide. But due to the singularity of $\Delta p(t, \cdot)$ near zero, we instead prove

$$\int_0^T \int_{\mathbf{R}^d} \partial_t^\alpha p(t, x) h(t) \phi(x) dx dt = \int_0^T \int_{\mathbf{R}^d} \Delta p(t, x) h(t) \phi(x) dx dt \quad (6.37)$$

for any $\phi \in C_c^\infty(\mathbf{R}^d \setminus \{0\})$ and $h(t) \in C_c^\infty((0, T))$.

Note that $t^{-\alpha}|x|^2 \geq c > 0$ on the support of $h\phi$, and thus (3.2) implies that $\frac{\partial p}{\partial t}(t, x)$, $\partial_t^\alpha p(t, x)$, and $\Delta p(t, x)$ are bounded on the support of $h\phi$. Hence both sides of (6.37) make sense. By the integration by parts,

$$\int_0^T \partial_t^\alpha p(t, x) h(t) dt = \int_0^T p(T-t, x) D_t^\alpha H(t) dt,$$

where $H(t) := h(T-t)$. Recall that by Parseval's identity

$$\int_{\mathbf{R}^d} f \bar{g} dx = \int_{\mathbf{R}^d} (\mathcal{F}f)(\bar{\mathcal{F}}g) d\xi, \quad \forall f, g \in L_2(\mathbf{R}^d).$$

Considering an approximation of f by functions in $L_2(\mathbf{R}^d)$ one can easily prove that Parseval's identity holds if $f \in L_1(\mathbf{R}^d)$ and g is in the Schwartz class. Hence the left side of (6.37) equals

$$\begin{aligned} & \int_0^T \left[\int_{\mathbf{R}^d} p(T-t, x) \phi(x) dx \right] \partial_t^\alpha H(t) dt \\ &= \int_0^T \left[\int_{\mathbf{R}^d} E_\alpha(-(T-t)^\alpha |\xi|^2) \mathcal{F}(\phi)(\xi) d\xi \right] \partial_t^\alpha H(t) dt. \end{aligned}$$

Observe that $\mathcal{F}(\phi)(\xi) \rightarrow 0$ significantly fast as $|\xi| \rightarrow \infty$ and $\partial_t^\alpha H(t) = 0$ if t is sufficiently small. Hence by (3.1) we can apply Fubini's theorem and show that the last term above is equal to

$$\begin{aligned} & \int_{\mathbf{R}^d} \left[\int_0^T E_\alpha(-(T-t)^\alpha |\xi|^2) \partial_t^\alpha H(t) dt \right] \mathcal{F}(\phi)(\xi) d\xi \\ &= \int_{\mathbf{R}^d} \int_0^T E_\alpha(-t^\alpha |\xi|^2) h(t) \mathcal{F}(\Delta \phi)(\xi) dt d\xi \\ &= \int_0^T \int_{\mathbf{R}^d} p(t, x) h(t) \Delta \phi(x) dx dt = \int_0^T \int_{\mathbf{R}^d} \Delta p(t, x) h(t) \phi(x) dx dt. \end{aligned}$$

For the first equality above we use the integration by parts in time variable and the fact $\partial_t^\alpha E_\alpha(-t^\alpha |\xi|^2) = -|\xi|^2 E_\alpha(-t^\alpha |\xi|^2)$, and Parseval's identity is used for the second equality. Therefore the first assertion of (6.37) is proved.

Next we prove $\Delta q = \frac{\partial p}{\partial t}$. Note that due to (3.2), if $x \neq 0$ then $D_x p(\cdot, x)$ and $D_x^2 p(\cdot, x)$ are bounded on $(0, T)$ uniformly in a neighborhood of x . Hence,

if $\alpha \in (0, 1)$,

$$\begin{aligned}\Delta q &= \Delta \frac{d}{dt} \int_0^t k_\alpha(t-s)p(s, x)ds \\ &= \frac{d}{dt} \Delta \int_0^t k_\alpha(t-s)p(s, x)ds \\ &= \frac{d}{dt} \int_0^t k_\alpha(t-s)\Delta p(s, x)ds = D^{1-\alpha} \partial_t^\alpha p = \frac{\partial p}{\partial t}.\end{aligned}$$

Similarly, if $\alpha \in (1, 2)$,

$$\Delta q = \Delta(I_t^{\alpha-1}p) = I_t^{\alpha-1}(\Delta p) = I_t^{\alpha-1}\partial_t^\alpha p = \frac{\partial p}{\partial t}.$$

Thus the second assertion of (6.36) is proved.

REFERENCES

- [1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. Number 55. Courier Dover Publications, 1972.
- [2] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71. Cambridge University Press, 1999.
- [3] R. L. Bagley and P. Torvik. A theoretical basis for the application of fractional calculus to viscoelasticity. *Journal of Rheology (1978-present)*, 27(3):201–210, 1983.
- [4] D. A. Benson, S. W. Wheatcraft, and M. M. Meerschaert. Application of a fractional advection-dispersion equation. *Water Resources Research*, 36(6):1403–1412, 2000.
- [5] B. L. J. Braaksma. Asymptotic expansions and analytic continuations for a class of barnes-integrals. *Compositio Mathematica*, 15:239–341, 1936.
- [6] R. Caponetto. *Fractional order systems: modeling and control applications*, volume 72. World Scientific, 2010.
- [7] P. Clément, S.-O. Londen, and G. Simonett. Quasilinear evolutionary equations and continuous interpolation spaces. *Journal of Differential Equations*, 196(2):418–447, 2004.
- [8] P. Clément and J. Prüss. Global existence for a semilinear parabolic volterra equation. *Mathematische Zeitschrift*, 209(1):17–26, 1992.
- [9] G. Da Prato and M. Iannelli. Existence and regularity for a class of integrodifferential equations of parabolic type. *Journal of Mathematical Analysis and Applications*, 112(1):36–55, 1985.
- [10] M. M. Djrbashian. *Harmonic analysis and boundary value problems in the complex domain*, volume 65. Springer, 1993.
- [11] S. D. Eidelman, S. D. Ivasyshen, and A. N. Kochubei. *Analytic methods in the theory of differential and pseudo-differential equations of parabolic type*, volume 152. Springer, 2004.
- [12] S. D. Eidelman and A. N. Kochubei. Cauchy problem for fractional diffusion equations. *Journal of Differential Equations*, 199(2):211–255, 2004.
- [13] N. Engheia. On the role of fractional calculus in electromagnetic theory. *Antennas and Propagation Magazine, IEEE*, 39(4):35–46, 1997.
- [14] W. G. Glöckle and T. F. Nonnenmacher. A fractional calculus approach to self-similar protein dynamics. *Biophysical Journal*, 68(1):46–53, 1995.
- [15] R. Hilfer, P. Butzer, U. Westphal, J. Douglas, W. Schneider, G. Zaslavsky, T. Nonnenmacher, A. Blumen, and B. West. *Applications of fractional calculus in physics*, volume 5. World Scientific, 2000.
- [16] A. A. Kilbas. *H-transforms: Theory and Applications*. CRC Press, 2004.
- [17] I. Kim, K.-H. Kim, and S. Lim. Parabolic BMO estimates for pseudo-differential operators of arbitrary order. *arXiv preprint arXiv:1408.2343*, 2014.
- [18] A. N. Kochubei. Asymptotic properties of solutions of the fractional diffusion-wave equation. *Fractional Calculus and Applied Analysis*, 17(3):881–896, 2014.
- [19] N. V. Krylov. An analytic approach to SPDEs. *Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs*, 64:185–242, 1999.

- [20] N. V. Krylov. On the Calderón-Zygmund theorem with applications to parabolic equations. *Algebra i Analiz*, 13(4):1–25, 2001.
- [21] P. C. Kunstmann and L. Weis. Maximal L_p -regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞ -functional Calculus. In *Functional analytic methods for evolution equations*, pages 65–311. Springer, 2004.
- [22] T. Langlands, B. Henry, and S. Wearne. Fractional cable equation models for anomalous electrodiffusion in nerve cells: infinite domain solutions. *Journal of mathematical biology*, 59(6):761–808, 2009.
- [23] F. Mainardi. Fractional diffusive waves in viscoelastic solids. *Nonlinear Waves in Solids, Fairfield*, pages 93–97, 1995.
- [24] M. M. Meerschaert and A. Sikorskii. *Stochastic models for fractional calculus*, volume 43. Walter de Gruyter, 2011.
- [25] R. Metzler, E. Barkai, and J. Klafter. Anomalous diffusion and relaxation close to thermal equilibrium: a fractional fokker-planck equation approach. *Physical Review Letters*, 82(18):3563, 1999.
- [26] R. Metzler and J. Klafter. Boundary value problems for fractional diffusion equations. *Physica A: Statistical Mechanics and its Applications*, 278(1):107–125, 2000.
- [27] R. Metzler and J. Klafter. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. *Physics reports*, 339(1):1–77, 2000.
- [28] R. Metzler and J. Klafter. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *Journal of Physics A: Mathematical and General*, 37(31):R161, 2004.
- [29] R. Metzler, W. Schick, H.-G. Kilian, and T. F. Nonnenmacher. Relaxation in filled polymers: A fractional calculus approach. *The Journal of Chemical Physics*, 103(16):7180–7186, 1995.
- [30] M. D. Ortigueira. *Fractional calculus for scientists and engineers*, volume 84. Springer, 2011.
- [31] I. Podlubny. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, volume 198. Academic press, 1998.
- [32] I. Podlubny. Fractional-order systems and $PI^\lambda D^\mu$ -controllers. *IEEE Transactions on Automatic Control*, 44(1):208–214, 1999.
- [33] J. Prüss. Quasilinear parabolic Volterra equations in spaces of integrable functions. In *B. de Pagter, Ph. Clément, E. Mitidieri, editors, Semigroup Theory and Evolution Equations, Lecture Notes in Pure and Applied Mathematics*, 135:401–420, 1991.
- [34] J. Prüss. *Evolutionary integral equations and applications*. Springer, 2012.
- [35] A. V. Pskhu. The fundamental solution of a diffusion-wave equation of fractional order. *Izvestiya: Mathematics*, 73(2):351, 2009.
- [36] M. Raberto, E. Scalas, and F. Mainardi. Waiting-times and returns in high-frequency financial data: an empirical study. *Physica A: Statistical Mechanics and its Applications*, 314(1):749–755, 2002.
- [37] J. Sabatier, O. P. Agrawal, and J. T. Machado. *Advances in fractional calculus*. Springer, 2007.
- [38] K. Sakamoto and M. Yamamoto. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *Journal of Mathematical Analysis and Applications*, 382(1):426–447, 2011.
- [39] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional integrals and derivatives*, volume 1993. 1993.
- [40] E. Scalas, R. Gorenflo, and F. Mainardi. Fractional calculus and continuous-time finance. *Physica A: Statistical Mechanics and its Applications*, 284(1):376–384, 2000.
- [41] R. Schumer, D. A. Benson, M. M. Meerschaert, and S. W. Wheatcraft. Eulerian derivation of the fractional advection–dispersion equation. *Journal of Contaminant Hydrology*, 48(1):69–88, 2001.
- [42] E. M. Stein and G. L. Weiss. *Introduction to Fourier analysis on Euclidean spaces*, volume 1. Princeton university press, 1971.
- [43] V. E. Tarasov. Electromagnetic fields on fractals. *Modern Physics Letters A*, 21(20):1587–1600, 2006.
- [44] L. von Wolfersdorf. On identification of memory kernels in linear theory of heat conduction. *Mathematical methods in the applied sciences*, 17(12):919–932, 1994.

- [45] H. Ye, J. Gao, and Y. Ding. A generalized Gronwall inequality and its application to a fractional differential equation. *Journal of Mathematical Analysis and Applications*, 328(2):1075–1081, 2007.
- [46] R. Zacher. Maximal regularity of type L_p for abstract parabolic Volterra equations. *Journal of Evolution Equations*, 5(1):79–103, 2005.
- [47] R. Zacher. Weak solutions of abstract evolutionary integro-differential equations in hilbert spaces. *Funkcialaj Ekvacioj*, 52(1):1–18, 2009.
- [48] G. M. Zaslavsky. Chaos, fractional kinetics, and anomalous transport. *Physics Reports*, 371(6):461–580, 2002.

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